# DETECTING NONTRIVIAL PRODUCTS IN THE STABLE HOMOTOPY RING OF SPHERES VIA THE THIRD MORAVA STABILIZER ALGEBRA

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Abstract. Let $p \geq 7$ be a prime number. Let $S(3)$ denote the third Morava stabilizer algebra. In recent years, Kato-Shimomura and Gu-Wang-Wu found several families of nontrivial products in the stable homotopy ring of spheres $\pi_*(S)$ using $H^{*,*}(S(3))$. In this paper, we determine all nontrivial products in $\pi_*(S)$ of the Greek letter family elements $\alpha_s, \beta_s, \gamma_s$ and Cohen's elements $\zeta_n$ which are detectable by $H^{*,*}(S(3))$. In particular, we show $\beta_1\gamma_s\zeta_n \neq 0 \in \pi_*(S)$, if $n \equiv 2 \mod 3$, $s \not\equiv 0, \pm 1 \mod p$.

## 1. Introduction

The computation of the ring of stable homotopy groups of spheres, denoted as $\pi_*(S)$, is one of the fundamental problems in algebraic topology. The Adams-Novikov spectral sequence (ANSS) based on the Brown-Peterson spectrum $BP$ is an incredibly powerful tool for computing the $p$-component of $\pi_*(S)$, where $p$ is a prime number. The $E_2$-page of the ANSS is of the form $Ext^{s,t}_{BP_*BP}(BP_*, BP_*)$ and has been extensively studied in low dimensions.

For $s = 1$, $Ext^{1,*}_{BP_*BP}(BP_*, BP_*)$ is generated by $\alpha_{kp^n/n+1}$ for $n \geqslant 0$, and $p \nmid k \geqslant 1$ ([15]).

For $s = 2$, $Ext^{2,*}_{BP_*BP}(BP_*, BP_*)$ is generated by $\beta_{kp^n/j,i+1}$ for suitable $(n, k, j, i)$ ([11, 12]).

For $s = 3$, only partial results of $Ext^{3,*}_{BP_*BP}(BP_*, BP_*)$ are known (see, for example, [13, 14, 18]). Nonetheless, a construction of a family of linearly independent elements denoted as $\gamma_{s_3/s_2,s_1}$ in $Ext^{3,*}_{BP_*BP}(BP_*, BP_*)$ has been achieved ([11]).

Through the computations of $Ext^{s,t}_{BP_*BP}(BP_*, BP_*)$ in low dimensions, numerous nontrivial elements in $\pi_*(S)$ can be obtained. In particular, for $p \geq 7$, there are the Greek letter family elements, denoted as $\alpha_s$, $\beta_s$, and $\gamma_s$ with $s \geq 1$ [11, 15, 19, 20]. These families are represented by elements of the same name in $Ext^{1,*}_{BP_*BP}(BP_*, BP_*)$, $Ext^{2,*}_{BP_*BP}(BP_*, BP_*)$, and $Ext^{3,*}_{BP_*BP}(BP_*, BP_*)$, respectively.

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Furthermore, using the Adams spectral sequence, Cohen [2] discovered another family of nontrivial elements $\zeta_n \in \pi_*(S)$ with $n \geq 1$. The representation of $\zeta_n$ in $Ext^{3,*}_{BP_*BP}(BP_*, BP_*)$ has also been studied in [2] (also see [3]).

Nontrivial products on $\pi_*(S)$. There exists a natural ring structure on $\pi_*(S)$ in which multiplication is defined by the composition of representing maps. In order to gain a deeper understanding of the ring structure of $\pi_*(S)$, it is necessary to determine whether the product of certain given elements is trivial. The main purpose of this paper is to find nontrivial products formed by the elements in $\{\alpha_s, \beta_s, \gamma_s, \zeta_s | s \geq 1\}$. To ensure these elements are well-defined, we assume $p \geq 7$ for the remainder of the paper, unless otherwise specified.

Numerous results have been obtained in this direction. Just to mention a few:

(a) Aubry [1] shows that $\alpha_1\beta_2\gamma_2, \beta_1^r\beta_2\gamma_2 \neq 0$ if $r \leq p-1$.
(b) Lee-Ravenel [8] shows $\beta_1^{p^2-p-1} \neq 0$ for $p \geq 7$.
(c) Lee [7] shows: (1) $\beta_1^r\beta_s, \beta_1^{r-1}\beta_2\beta_{kp-1} \neq 0$ for $p \geq 5$, if $r, k \leq p-1$, $s < p^2-p-1$, and $s \not\equiv 0 \bmod p$; (2) $\beta_1^r\gamma_t, \beta_1^{r-1}\beta_2\gamma_t \neq 0$, if $r, t \leq p-1$; (3) $\alpha_1\beta_1^r\gamma_t \neq 0$, if $r \leq p-2$, $2 \leq t \leq p-1$; (4) $\beta_1^{p-1}\zeta_n \neq 0$.
(d) Liu-Liu [9] shows that $\alpha_1\beta_1^2\beta_2\gamma_s \neq 0$ if $4 < s < p$.
(e) Zhao-Wang-Zhong [23] shows that $\gamma_{p-1}\zeta_n \neq 0$ if $n \neq 4$.

In recent years, Kato-Shimomura [5] have developed a method for detecting nontrivial products on $\pi_*(S)$ through the use of $S(3)$, where $S(3)$ denotes the third Morava stabilizer algebra [16]. This new approach offers an advantage when studying products involving $\gamma_s$ for arbitrarily large values of $s$. We can briefly recall their strategy as follows.

There exists a natural map $\phi : Ext^{*,*}_{BP_*BP}(BP_*, BP_*) \to Ext^{*,*}_{S(3)}(\mathbb{F}_p, \mathbb{F}_p) =: H^{*,*}(S(3))$. The cohomology $H^{*,*}(S(3))$ is studied in [3, 17, 22]. Given a product $x = x_1x_2\cdots x_n \in \pi_*(S)$, we let $y = y_1y_2\cdots y_n \in Ext^{*,*}_{BP_*BP}(BP_*, BP_*)$ represent $x$ on the $E_2$-page of the ANSS. If $\phi(y) \neq 0$, then $y \neq 0 \in Ext^{*,*}_{BP_*BP}(BP_*, BP_*)$. For the examples of interest, $y$ will not be eliminated by any Adams-Novikov differential due to degree considerations. Thus, we can conclude that $x \neq 0 \in \pi_*(S)$ in this case.

Using this strategy, Kato-Shimomura [5] demonstrate the following: (1) $\alpha_1\gamma_s \neq 0$, if $s \not\equiv 0, \pm 1 \bmod p$; (2) $\beta_1\gamma_s \neq 0$, if $s \not\equiv 0, 1 \bmod p$; (3) $\beta_2\gamma_s \neq 0$, if $s \not\equiv 0, \pm 1 \bmod p$.

Similarly, Gu-Wang-Wu [3] show that $\zeta_n\gamma_s \neq 0$ if $n \not\equiv 1 \bmod 3$ and $s \not\equiv 0, \pm 1 \bmod p$.

Our main results. In this paper, we employ the "Detection via $H^{*,*}(S(3))$" method, which was developed in [3, 5], to detect nontrivial products on $\pi_*(S)$. However, instead of focusing on specific examples, we fully utilize the potential of this method and enumerate all detectable products. The main results of our study are as follows:

Theorem 1.1. Let $p \geq 7$ be a prime. Let $n \equiv 2 \bmod 3$, and $s \not\equiv 0, \pm 1 \bmod p$. Then $\beta_1\gamma_s\zeta_n \neq 0 \in \pi_*(S)$.

Remark 1.2. Utilizing the Adams spectral sequence, Kato-Shimomura [6] demonstrated that $\beta_1\gamma_s\zeta_n \neq 0 \in \pi_*(S)$ holds true when $3 \leq s < p-2$. The findings presented in [6] and Theorem 1.1 address distinct ranges of $(n, s)$, with neither being a subset of the other. The method of "Detection via $H^{*,*}(S(3))$" possesses the advantage of accommodating products involving $\gamma_s$ for arbitrarily large $s$.

Theorem 1.3. Let $p \geq 7$ be a prime. We consider the products in $\pi_*(S)$ where each factor belongs to $\{\alpha_s, \beta_s, \gamma_s, \zeta_s : s \geq 1\}$. Among all such products, only the following ones can be detected as nontrivial products using the comparison with $H^{*,*}S(3)$.

(1) $\alpha_1\beta_1$,
(2) $\alpha_1\beta_2$,
(3) $\alpha_1\gamma_s$, if $s \not\equiv 0, \pm 1 \bmod p$,
(4) $\beta_1^2$,
(5) $\beta_1\gamma_s$, if $s \not\equiv 0, 1 \bmod p$,
(6) $\beta_1\zeta_n$,
(7) $\beta_2\gamma_s$, if $s \not\equiv 0, \pm 1 \bmod p$,
(8) $\gamma_s\zeta_n$, if $n \not\equiv 1 \bmod 3$, $s \not\equiv 0, \pm 1 \bmod p$,
(9) $\alpha_1\beta_1^2$,
(10) $\alpha_1\beta_1\gamma_s$, if $s \not\equiv 0, \pm 1 \bmod p$,
(11) $\beta_1^2\zeta_n$, if $n \equiv 1 \bmod 3$
(12) $\beta_1\gamma_s\zeta_n$, if $n \equiv 2 \bmod 3$, $s \not\equiv 0, \pm 1 \bmod p$.

The non-triviality of (1) $\sim$ (11) has been determined by earlier works in [3, 5, 7, 11]. We single out the new result (12) as Theorem 1.1. We have exhausted the potential of the "Detection via $H^{*,*}(S(3))$" strategy in Theorem 1.3. To detect other nontrivial products in $\pi_*(S)$, different methods would need to be employed.

Organization of the paper. In Section 2, we review the basic structures of the Hopf algebroid $(BP_*, BP_*BP)$ and the third Morava stabilizer algebra $S(3)$. In Section 3, we analyze the $\mathbb{F}_p$-algebra structure of $H^{*,*}S(3)$. We also discuss some typos in the previous literature [3, 22]. In Section 4, we determine the images of $\{\alpha_s, \beta_s, \gamma_s, \zeta_s | s \geq 1\}$ under the comparison map $\phi : Ext^{*,*}_{BP_*BP}(BP_*, BP_*) \to H^{*,*}(S(3))$. In Section 5, we prove Theorem 1.1 and Theorem 1.3.

## 2. Hopf algebroids

This section recalls the basic definitions and constructions related to Hopf algebroids. In particular, we review the basic structures of the Hopf algebroid $(BP_*, BP_*BP)$ and the third Morava stabilizer algebra $S(3)$.

### 2.1. The Hopf algebroid $(BP_*, BP_*BP)$.

Definition 2.2. A Hopf algebroid over a commutative ring $K$ is a pair $(A, \Gamma)$ of commutative $K$-algebras with structure maps

$$\begin{aligned} \text{left unit map } \eta_L &: A \to \Gamma \\ \text{right unit map } \eta_R &: A \to \Gamma \\ \text{coproduct map } \Delta &: \Gamma \to \Gamma \otimes_A \Gamma \\ \text{counit map } \varepsilon &: \Gamma \to A \\ \text{conjugation map } c &: \Gamma \to \Gamma \end{aligned}$$

such that for any other commutative $K$-algebra $B$, the two sets $\mathrm{Hom}(A, B)$ and $\mathrm{Hom}(\Gamma, B)$ are the objects and morphisms of a groupoid.

An important example of Hopf algebroids is $(BP_*, BP_*BP)$. Recall that we have

$$BP_* := \pi_*(BP) = \mathbb{Z}_{(p)}[v_1, v_2, \cdots], \quad BP_*BP = BP_*[t_1, t_2, \cdots] \tag{2.1}$$

where the inner degrees are $|v_n| = |t_n| = 2(p^n - 1)$. Throughout this paper, we denote $v_0 = p$, and $t_0 = 1$. The structure maps of the Hopf algebroid $(BP_*, BP_*BP)$ are described in [4, 11, 18]. In practice, the following formulas [5] are useful.

$$\eta_R(v_1) = v_1 + pt_1, \tag{2.2}$$

$$\eta_R(v_2) \equiv v_2 + v_1 t_1^p + pt_2 \mod (p^2, v_1^p), \tag{2.3}$$

$$\Delta(t_1) = t_1 \otimes 1 + 1 \otimes t_1, \tag{2.4}$$

$$\Delta(t_2) = t_2 \otimes 1 + t_1 \otimes t_1^p + 1 \otimes t_2 - v_1 b_{1,0}. \tag{2.5}$$

Notations 2.3. We denote $b_{i,j} = \frac{1}{p}[(\sum_{k=0}^{i} t_{i-k} \otimes t_k^{p^{i-k}})^{p^{j+1}} - \sum_{k=0}^{i} t_{i-k}^{p^{j+1}} \otimes t_k^{p^{i-k+j+1}}]$ for $i \geq 1$, $j \geq 0$. See [21] for related discussions.

2.4. Morava stabilizer algebras. We recall the basic properties of the Morava stabilizer algebras, which are studied in detail in [10, 16].

Let $K(n)_*$ denote $\mathbb{F}_p[v_n, v_n^{-1}]$. We can equip $K(n)_*$ with a $BP_*$-algebra structure via the ring homomorphism which sends all $v_i$ with $i \neq n$ to 0. Then we define $\Sigma(n) := K(n)_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} K(n)_*$. As an algebra, one has $\Sigma(n) \cong K(n)_*[t_1, t_2, \cdots]/(v_n t_i^{p^n} - v_n^{p^i} t_i | i > 0)$. The coproduct structure of $\Sigma(n)$ is inherited from that of $BP_*BP$.

Moreover, one can prove $Ext^{*,*}_{BP_*BP}(BP_*, v_n^{-1}BP_*/I_n) \cong Ext^{*,*}_{\Sigma(n)}(K(n)_*, K(n)_*)$, where we let $I_n$ denote the ideal $(p, v_1, v_2, \cdots, v_{n-1}) \subset BP_*$.

We define the Hopf algebra $S(n) := \Sigma(n) \otimes_{K(n)_*} \mathbb{F}_p$, where $K(n)_*$ and $\Sigma(n)$ are here regarded as graded over $\mathbb{Z}/2(p^n - 1)$ and $\mathbb{F}_p$ is a $K(n)_*$-algebra via the map sending $v_n$ to 1. We call $S(n)$ the $n$-th Morava stabilizer algebra. One can show

$$Ext^{*,*}_{\Sigma(n)}(K(n)_*, K(n)_*) \otimes_{K(n)_*} \mathbb{F}_p \cong Ext^{*,*}_{S(n)}(\mathbb{F}_p, \mathbb{F}_p) =: H^{*,*}(S(n)) \tag{2.6}$$

For the purpose of this paper, from now on, we will only consider the case when $n = 3$. We have the following results.

Proposition 2.5 ([17]). As an algebra, $S(3) \cong \mathbb{F}_p[t_1, t_2, ...]/(t_i^{p^3} - t_i)$ and the inner degrees are $|t_s| \equiv 2(p^s - 1) \mod 2(p^3 - 1)$. The coproduct structure of $S(3)$ is that inherited from $BP_*BP$. In particular, $\Delta(t_s) = \sum_{k=0}^{s} t_k \otimes t_{s-k}^{p^k}$ for $s \leq 3$, and $\Delta(t_s) = \sum_{k=0}^{s} t_k \otimes t_{s-k}^{p^k} - \tilde{b}_{s-3,2}$ for $s > 3$.

Notations 2.6. We let $\tilde{b}_{i,j}$ denote the mod $p$ reduction of $b_{i,j}$ in Notations 2.3.

2.7. Cobar complexes. Cobar complexes are helpful in computing certain $Ext$ groups, such as $Ext^{*,*}_{BP_*BP}(BP_*, BP_*)$, $Ext^{*,*}_{BP_*BP}(BP_*, v_n^{-1}BP_*/I_n)$, and $Ext^{*,*}_{S(n)}(\mathbb{F}_p, \mathbb{F}_p)$. We now recall the relevant definitions and constructions.

Definition 2.8. Let $(A, \Gamma)$ be a Hopf algebroid. A right $\Gamma$-comodule $M$ is a right $A$-module $M$ together with a right $A$-linear map $\psi : M \to M \otimes_A \Gamma$ which is counitary and coassociative. Left $\Gamma$-comodules are defined similarly.

Definition 2.9. Let $(A, \Gamma)$ be a Hopf algebroid. Let $M$ be a right $\Gamma$-comodule. The cobar complex $\Omega^{*,*}_\Gamma(M)$ is a cochain complex with $\Omega^{s,*}_\Gamma(M) = M \otimes_A \overline{\Gamma}^{\otimes s}$, where $\overline{\Gamma}$ is the augmentation ideal of $\varepsilon : \Gamma \to A$. The differentials $d : \Omega^{s,*}_\Gamma(M) \to \Omega^{s+1,*}_\Gamma(M)$ are given by

$$d(m \otimes x_1 \otimes x_2 \otimes \cdots \otimes x_s) = -(\psi(m) - m \otimes 1) \otimes x_1 \otimes x_2 \otimes \cdots \otimes x_s$$
$$- \sum_{i=1}^{s} (-1)^{\lambda_{i,j_i}} m \otimes x_1 \otimes \cdots \otimes x_{i-1} \otimes (\sum_{j_i} x'_{i,j_i} \otimes x''_{i,j_i}) \otimes x_{i+1} \otimes \cdots \otimes x_s$$

where we denote

$$\sum_{j_i} x'_{i,j_i} \otimes x''_{i,j_i} = \Delta(x_i) - 1 \otimes x_i - x_i \otimes 1 \tag{2.7}$$

$$\lambda_{i,j_i} = i + |x_1| + \cdots + |x_{i-1}| + |x'_{i,j_i}|. \tag{2.8}$$

Proposition 2.10 ([18] Section A1.2). The cohomology of $\Omega_\Gamma^{s,*}(M)$ is $Ext_\Gamma^{s,*}(A, M)$. Moreover, if $M$ is also a commutative associative $A$-algebra such that the structure map $\psi$ is an algebra map, then $Ext_\Gamma^{s,*}(A, M)$ has a naturally induced product structure.

## 3. The cohomology of $S(3)$

In this section, we discuss the cohomology $H^{*,*}S(3) := Ext^{*,*}_{S(3)}(\mathbb{F}_p, \mathbb{F}_p)$ of the Hopf algebra $S(3)$. Ravenel [17] computed the $\mathbb{F}_p$-module structure of $H^{*,*}S(3)$. The $\mathbb{F}_p$-algebra structure of $H^{*,*}S(3)$ was subsequently computed by Yamaguchi in [22], and revisited by Gu-Wang-Wu in [3]. Unfortunately, both [22] and [3] contain typos. We will say more about these typos in this section.

Theorem 3.1 ([3, 22]). As an $\mathbb{F}_p$-algebra, $H^*(S(3))$ is isomorphic to the cohomology $H^*(E; d_1)$ of a certain differential graded algebra $E$, where

$$E := E[h_{i,j} | i = 1, 2, 3, j \in \mathbb{Z}/3], \tag{3.1}$$

is the exterior algebra with generators $h_{i,j}$, and the differential $d_1$ is given by

$$d_1(h_{i,j}) = -\sum_{1 \le k \le i-1} h_{k,j} h_{i-k,j+k}. \tag{3.2}$$

Moreover,

$$d_1(xy) = d_1(x)y + (-1)^s x d_1(y) \tag{3.3}$$

for all monomials $x, y \in E$ and $s$ denotes the homological degree of $x$. The generator $h_{i,j}$ corresponds to $t_i^{p^j} \in S(3)$ under the isomorphism $H^*(E; d_1) \cong H^*(S(3))$. The generator $h_{i,j}$ has induced inner degree $|h_{i,j}| = 2(p^i - 1)p^j \bmod 2(p^3 - 1)$.

Remark 3.2. Recall from Proposition 2.5 that $S(3) \cong \mathbb{F}_p[t_1, t_2, ...]/(t_i^{p^3} - t_i)$. This implies $t_i^{p^j} = t_i^{p^{j+3}} \in S(3)$. Corresponding to this, we have $j \in \mathbb{Z}/3$ in Theorem 3.1.

Proposition 3.3 ([3, 22]). Let $p \ge 7$ be a prime number. As a $\mathbb{F}_p$-module, $H^{*,*}S(3)$ is isomorphic to $E[\rho] \otimes M$, where $\rho := h_{3,0} + h_{3,1} + h_{3,2} \in H^{1,*}S(3)$, $M$ is a $\mathbb{F}_p$-module with the following generators ($i \in \mathbb{Z}/3$):

dim0: $1$;
dim1: $h_{1,i}$;
dim2: $e_{4,i}$, $g_i$, $k_i$;
dim3: $e_{4,i}h_{1,i}$, $e_{4,i}h_{1,i+1}$, $g_i h_{1,i+1}$, $\mu_i$, $\nu_i$, $\xi$;
dim4: $e_{4,i}^2$, $e_{4,i}e_{4,i+1}$, $e_{4,i}g_{i+1}$, $e_{4,i}g_{i+2}$, $e_{4,i}k_i$, $\theta_i$;
dim5: $e_{4,i}^2 h_{1,i+1}$, $e_{4,i}^2 h_{1,i+2}$, $e_{4,i}e_{4,i+1}h_{1,i+2}$ $(e_{4,i}e_{4,i+1}h_{1,i+2} = e_{4,i+1}e_{4,i+2}h_{1,i})$,
$e_{4,i}\mu_{i+2}$, $e_{4,i}\nu_i$, $\eta_i$;
dim6: $e_{4,i}^2 e_{4,i+1}$, $e_{4,i}^2 e_{4,i+2}$, $e_{4,i}e_{4,i+1}g_{i+2}$;
dim7: $e_{4,i}e_{4,i+1}\mu_{i+2}$;
dim8: $e_{4,i}^2 e_{4,i+2}g_{i+1}$ $(e_{4,i}^2 e_{4,i+2}g_{i+1} = e_{4,i+1}^2 e_{4,i}g_{i+2})$.

Here, the generators are defined as follows:

$$
\begin{array}{rlrl}
e_{4,i} & := h_{1,i}h_{3,i+1} + h_{2,i}h_{2,i+2} + h_{3,i}h_{1,i} & g_i & := h_{2,i}h_{1,i} \\
k_i & := h_{2,i}h_{1,i+1} & \mu_i & = h_{3,i}h_{2,i}h_{1,i} \\
\nu_i & := h_{3,i}h_{2,i+1}h_{1,i+2} & \xi & = \sum_{i=0}^{2} h_{3,i}e_{3,i+1} + h_{2,0}h_{2,1}h_{2,2} \\
\theta_i & = h_{3,i}h_{2,i+2}h_{2,i}h_{1,i} & \eta_i & = h_{3,i}h_{3,i+1}h_{2,i+2}h_{2,i}h_{1,i}
\end{array}
$$

Here we denote $e_{3,i} := h_{1,i}h_{2,i+1} + h_{2,i}h_{1,i+2}$ for $i \in \mathbb{Z}/3$.

Remark 3.4. The original formula for $\xi$ in [3] was $\xi = \sum h_{3,i+1}e_{3,i} + \sum h_{2,i}h_{2,i+1}h_{2,i+2}$. However, that doesn't represent a cocycle. We have corrected the formula for $\xi$ in Proposition 3.3. It corresponds to Yamaguchi's generator $c$ in [22] under the relation $c - \xi = -d_1(h_{3,0}h_{3,2})$.

Using Theorem 3.1, one can compute the product relations of these additive generators by hand.

Example 3.5. Direct computation shows

(1) $h_{1,i}k_i = h_{1,i}h_{2,i}h_{1,i+1} = -h_{2,i}h_{1,i}h_{1,i+1} = -g_ih_{1,i+1}$.
(2) $e_{4,i+1}k_i = h_{1,i+1}h_{3,i+2}h_{2,i}h_{1,i+1} + h_{2,i+1}h_{2,i}h_{2,i}h_{1,i+1} + h_{3,i+1}h_{1,i+1}h_{2,i}h_{1,i+1} = 0$.
(3) $k_i^2 = h_{2,i}h_{1,i+1}h_{2,i}h_{1,i+1} = 0$.

It is also useful to notice that, for $x \in H^{i,*}S(3)$, $y \in H^{j,*}S(3)$, we have $x \cdot y = (-1)^{ij}y \cdot x$.

Computing the entire $\mathbb{F}_p$-algebra structure of $H^{*,*}S(3)$ is straightforward but quite tedious. Yamaguchi [22] and Gu-Wang-Wu [3] both listed the product relations without providing proofs. Unfortunately, both papers contain typos. As pointed out in [3, Remark A.1], the formula $a_0g_0' = h_0b_0' - h_1b_0$ in [22, Theorem 4.4] should be corrected to $a_0g_0' = h_0b_0' - 2h_1b_0$ under Yamaguchi's notation. On the other hand, [3, Appendix A] claimed $e_{4,i}e_{4,i+1}\nu_i = -e_{4,i}e_{4,i+1}\mu_{i+2} + \frac{2}{3}\rho e_{4,i+2}e_{4,i}g_{i+1}$. However, one can tell this is wrong since $e_{4,i}e_{4,i+1}\mu_{i+2}$ and $\rho e_{4,i+2}e_{4,i}g_{i+1}$ have different inner degrees. As another example, [3, Appendix A] claimed that $\nu_ih_{1,i} = \frac{1}{3}e_{4,i+1}g_{i+2}$ and $h_{1,i}e_{4,i+1}\nu_i = 0$. However, we have $h_{1,i}e_{4,i+1}\nu_i = h_{1,i}\nu_ie_{4,i+1} = -\frac{1}{3}e_{4,i+1}g_{i+2}e_{4,i+1} = -\frac{1}{3}e_{4,i+1}^2g_{i+2} \neq 0$, since $e_{4,i+1}^2e_{4,i}g_{i+2} \neq 0$ is a generator in dimension 8. This brings to a contradiction.

We have not reproduced all the product relations in $H^{*,*}S(3)$. We do not claim we have found all typos in [3, 22].

For the purpose of this paper, we do not attempt to determine the entire $\mathbb{F}_p$-algebra structure of $H^{*,*}S(3)$. In Proposition 3.6, we will recompute only the products that we actually need in this paper. Therefore, the result of this paper does not depend on the computation of the full $\mathbb{F}_p$-algebra structure of $H^{*,*}S(3)$ in [3, 22].

Proposition 3.6. Let $p \geq 7$ be a prime number. We have the following nontrivial products among generators of $H^{*,*}S(3)$:

- $\nu_0h_{1,0} = \frac{1}{3}e_{4,1}g_2 \neq 0$.
- $e_{4,1}\nu_0h_{1,0}e_{4,0} = \frac{1}{3}e_{4,1}^2e_{4,0}g_2 \neq 0$.
- $\nu_0e_{4,1} = -e_{4,2}\mu_1 + \frac{1}{3}\rho e_{4,2}g_1 + \frac{1}{3}\rho e_{4,1}k_1 \neq 0$.
- $e_{4,i+1}h_{1,i} = e_{4,i}h_{1,i+1} \neq 0$.
- $k_0\nu_0 = \frac{1}{2}e_{4,1}^2h_{1,2} \neq 0$.

Meanwhile, the following products are trivial:

$$h_{1,0}k_1 = 0,\ k_0k_1 = 0,\ k_0h_{1,0}e_{4,i} = 0,\ h_{1,0}k_0\nu_0 = 0$$

Proof. We pick several typical examples to illustrate the method of computation. The rest of the products can be computed similarly.

(1) $v_0h_{1,0} = \frac{1}{3}e_{4,1}g_2 \neq 0$. Recall that by definition, we have

$$v_0h_{1,0} = [h_{3,0}h_{2,1}h_{1,2}h_{1,0}], \tag{3.4}$$

$$e_{4,1}g_2 = [h_{1,1}h_{3,2}h_{2,2}h_{1,2} + h_{2,1}h_{2,0}h_{2,2}h_{1,2} + h_{3,1}h_{1,1}h_{2,2}h_{1,2}]. \tag{3.5}$$

Here, we use the bracket [ ] to emphasize these are cohomology classes. To simplify notations, we denote $A := h_{3,0}h_{2,1}h_{1,2}h_{1,0}$, $B := h_{1,1}h_{3,2}h_{2,2}h_{1,2}$, $C := h_{2,1}h_{2,0}h_{2,2}h_{1,2}$, and $D := h_{3,1}h_{1,1}h_{2,2}h_{1,2}$. We want to show $3[A] = [B + C + D]$. We also denote $E := h_{1,0}h_{2,1}h_{3,1}h_{1,2}$, $F := h_{3,0}h_{1,1}h_{2,2}h_{1,2}$, and $G := h_{2,1}h_{1,0}h_{3,2}h_{1,2}$.

Next, we consider elements in homological degree 3, which has the same inner degree as $A, B, C, D$. The point is, the differential of such elements might provide relations among $A, B, C, D$. Direct computation shows:

$$d_1(h_{3,0}h_{3,1}h_{1,2}) = E - F + A \Rightarrow [A] = [F] - [E]. \tag{3.6}$$

$$d_1(h_{3,0}h_{2,1}h_{2,2}) = -C + F + A \Rightarrow [A] = [C] - [F]. \tag{3.7}$$

$$d_1(h_{3,1}h_{3,2}h_{1,2}) = -B + G + D \Rightarrow [G] = [B] - [D]. \tag{3.8}$$

$$d_1(h_{3,0}h_{3,2}h_{1,2}) = -G + F \Rightarrow [F] = [G]. \tag{3.9}$$

$$d_1(h_{3,1}h_{2,1}h_{2,2}) = D + E \Rightarrow [E] = -[D]. \tag{3.10}$$

Then we have $3[A] = 2([F] - [E]) + ([C] - [F]) = [C] - 2[E] + [F] = [C] + 2[D] + [G] = [C] + 2[D] + [B] - [D] = [B] + [C] + [D]$. This shows $v_0h_{1,0} = \frac{1}{3}e_{4,1}g_2$. Moreover, $e_{4,1}g_2 \neq 0$ since it is a generator in dimension 4 by Proposition 3.3.

(2) $e_{4,1}v_0h_{1,0}e_{4,0} = \frac{1}{3}e_{4,1}^2e_{4,0}g_2 \neq 0$. This follows directly from (1). Moreover, $e_{4,1}^2e_{4,0}g_2 \neq 0$ since it is a generator in dimension 8 by Proposition 3.3.

(3) $h_{1,0}k_1 = [h_{1,0}h_{2,1}h_{1,2}] = 0$. This is because $d_1(h_{3,0}h_{1,2}) = h_{1,0}h_{2,1}h_{1,2} + h_{2,0}h_{1,2}h_{1,2} = h_{1,0}h_{2,1}h_{1,2}$.

(4) $k_0k_1 = [h_{2,0}h_{1,1}h_{2,1}h_{1,2}] = 0$. This is because $d_1(h_{3,0}h_{1,1}h_{2,1}) = h_{1,0}h_{2,1}h_{1,1}h_{2,1} + h_{2,0}h_{1,2}h_{1,1}h_{2,1} + h_{3,0}h_{1,1}h_{1,1}h_{1,2} = h_{2,0}h_{1,2}h_{1,1}h_{2,1}$.

The rest of the products can be computed similarly. ◻

## 4. Images of $\alpha, \beta, \gamma, \zeta$-family elements

In this section, we recall the constructions of the Greek letter family elements in the $E_2$-page $Ext^{*,*}_{BP_*BP}(BP_*, BP_*)$ of the Adams-Novikov spectral sequence. Then we determine the images of $\{\alpha_s, \beta_s, \gamma_s, \zeta_s | s \geq 1\}$ under the comparison map $\phi : Ext^{*,*}_{BP_*BP}(BP_*, BP_*) \to H^{*,*}(S(3))$.

Note we can write $\phi$ as the composition of several maps. We have

$$\phi = Ext^{*,*}_{BP_*BP}(BP_*, BP_*) \xrightarrow{\eta} Ext^{*,*}_{BP_*BP}(BP_*, v_3^{-1}BP_*/I_3) \xrightarrow{\psi} H^{*,*}(S(3)) \tag{4.1}$$

with $I_3 = (p, v_1, v_2) \subset BP_*$ and $\psi = \psi_3\psi_2\psi_1$, where

$$\psi_1 : Ext^{*,*}_{BP_*BP}(BP_*, v_3^{-1}BP_*/I_3) \xrightarrow{\cong} Ext^{*,*}_{\Sigma(3)}(K(3)_*, K(3)_*), \tag{4.2}$$

$$\psi_2 : Ext^{*,*}_{\Sigma(3)}(K(3)_*, K(3)_*) \to Ext^{*,*}_{\Sigma(3)}(K(3)_*, K(3)_*) \otimes_{K(3)_*} \mathbb{F}_p, \tag{4.3}$$

$$\psi_3 : Ext^{*,*}_{\Sigma(3)}(K(3)_*, K(3)_*) \otimes_{K(3)_*} \mathbb{F}_p \xrightarrow{\cong} Ext^{*,*}_{S(3)}(\mathbb{F}_p, \mathbb{F}_p) = H^{*,*}(S(3)). \tag{4.4}$$

4.1. $\alpha$-family elements. Let $n \geq 0$, $p \nmid s \geq 1$. Then $v_1^{sp^n} \in Ext^{0,*}_{BP_*BP}(BP_*, BP_*/p^{n+1})$. We define $\alpha_{sp^n/n+1} := \delta_0(v_1^{sp^n}) \in Ext^{1,*}_{BP_*BP}(BP_*, BP_*)$, where $\delta_0$ is the boundary-homomorphism associated to the short exact sequence

$$0 \to \Omega_{BP_*BP}(BP_*) \xrightarrow{p^{n+1}} \Omega_{BP_*BP}(BP_*) \to \Omega_{BP_*BP}(BP_*/p^{n+1}) \to 0 \tag{4.5}$$

of cobar complexes (Definition 2.9). We often abbreviate $\alpha_{s/1}$ to $\alpha_s$.

In order to determine the image of $\eta$, we introduce the following notion.

Definition 4.2. Let $n \geq 1$. We define $I[n]$ as the ideal of $BP_*$ generated by monomials $p^i v_1^j v_2^k$ such that $i + j + k = n$. In particular, $I[1] = (p, v_1, v_2) = I_3 \subset BP_*$.

Lemma 4.3. Let $d$ denote the differential of the cobar complex $\Omega^{*,*}_{BP_*BP}(BP_*)$. Let $x \in I[n] \subset BP_* = \Omega^{0,*}_{BP_*BP}(BP_*)$ for some $n \geq 1$. Then $d(x) \in I[n] \cdot \Omega^{1,*}_{BP_*BP}(BP_*)$.

Proof. $BP_*$ can be regarded as a right $BP_*BP$-comodule with $\eta_R : BP_* \to BP_*BP$ as the structure map. According to Definition 2.9, for $x \in BP_*$, we have

$$d(x) = -\psi(x) + x \otimes 1 = -\eta_R(x) + x \otimes 1. \tag{4.6}$$

Note that if $x \in I[n]$, then $x \otimes 1 \in I[n] \cdot \Omega^{1,*}_{BP_*BP}(BP_*)$. Therefore, it is sufficient to show that $\eta_R(x) \in I[n] \cdot \Omega^{1,*}_{BP_*BP}(BP_*)$. Furthermore, by considering each summand separately, we can assume that $x$ is a monomial in $BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \cdots]$. Write $x = p^i v_1^j v_2^k y$, where $i + j + k \geq n$. Using (2.2) and (2.3), we have

$$\begin{aligned} \eta_R(p^i v_1^j v_2^k y) &= \eta_R(p^i)\eta_R(v_1^j)\eta_R(v_2^k)\eta_R(y) \\ &= p^i(v_1 + pt_1)^j(v_2 + v_1 t_1^p + pt_2 + L)^k \eta_R(y) \end{aligned} \tag{4.7}$$

where $L \in (p^2, v_1^p) \cdot \Omega^{1,*}_{BP_*BP}(BP_*)$. By counting the exponents, we can see that $\eta_R(x) \in I[n] \cdot \Omega^{1,*}_{BP_*BP}(BP_*)$.

□

Proposition 4.4. Concerning the image of the $\alpha$-family elements under the map $\phi$ specified in (4.1), we have

(1) $\phi(\alpha_1) = -h_{1,0}$.
(2) $\phi(\alpha_s) = 0$, for $s > 1$.

Proof. (1) The image of $\alpha_1$ is computed in [5, Lemma 3.4]. Here, we still provide a detailed computation to illustrate the method.

We have $\alpha_1 = \delta_0(v_1)$. By definition of the connecting homomorphism $\delta_0$, we have

$$\delta_0(v_1) = \frac{d(v_1)}{p} = -\frac{(\eta_R(v_1) - v_1 \otimes 1)}{p} = -t_1, \tag{4.8}$$

where we let $v_1$ also denote the preimage of $v_1$ with respect to the map $\Omega_{BP_*BP}(BP_*) \to \Omega_{BP_*BP}(BP_*/p^{n+1})$ and let $d$ denote the differential of the cobar complex $\Omega_{BP_*BP}(BP_*)$. Therefore, upon reduction modulo $I_3$, we find that $\eta(\alpha_1) = -t_1$.

On the level of cobar complexes, the effect of $\psi$ is sending $v_3$ to 1. By Proposition 3.3, $-t_1$ represents $-h_{1,0}$ in $H^{*,*}(S(3))$. Therefore, $\phi(\alpha_1) = \psi(-t_1) = -h_{1,0}$.

(2) For $s \geq 2$, we have

$$\alpha_s = \delta_0(v_1^s) = \frac{d(v_1^s)}{p}. \tag{4.9}$$

Note that $v_1^s \in I[s]$. By Lemma 4.3, $d(v_1^s) \in I[s] \cdot \Omega_{BP_*BP}(BP_*)$. Then

$$\frac{d(v_1^s)}{p} \in I[s-1] \cdot \Omega_{BP_*BP}(BP_*). \tag{4.10}$$

Note $s-1 \geq 1$, we have

$$\alpha_s \in I[s-1] \cdot \Omega_{BP_*BP}(BP_*) \subset I[1] \cdot \Omega_{BP_*BP}(BP_*) = I_3 \cdot \Omega_{BP_*BP}(BP_*). \tag{4.11}$$

Upon reduction modulo $I_3$, we have $\eta(\alpha_s) = 0$. Therefore, $\phi(\alpha_s) = 0$ for $s > 1$. □

Notations 4.5. In this paper, we often abuse the notation and refer to the elements in $Ext_\Gamma^{s,*}(A, M)$ by their representatives in the associated cobar complex $\Omega_\Gamma^{s,*}(M)$ when no confusion arises. For example, here we let $-t_1$ denote the element in $Ext^{1,*}_{BP_*BP}(BP_*, v_3^{-1}BP_*/I_3)$ represented by $-t_1 \in \Omega^{1,*}_{BP_*BP}(v_3^{-1}BP_*/I_3)$.

Remark 4.6. Here, the result for $\phi(\alpha_1)$ differs from the formula in [5, Lemma 3.4] by a negative sign, as our definitions of the differential in the cobar complex (Definition 2.9) differ by a negative sign.

4.7. $\beta$-family elements. Let $a_0 = 1$, $a_n = p^n + p^{n-1} - 1$ for $n \geq 1$. Define $x_n \in v_2^{-1}BP_*$ as

$$x_0 = v_2, \tag{4.12}$$

$$x_1 = x_0^p - v_1^p v_2^{-1} v_3, \tag{4.13}$$

$$x_2 = x_1^p - v_1^{p^2-1} v_2^{p^2-p+1} - v_1^{p^2+p-1} v_2^{p^2-2p} v_3, \tag{4.14}$$

$$x_n = x_{n-1}^p - 2v_1^{b_n} v_2^{p^n-p^{n-1}+1}, n \geq 3 \tag{4.15}$$

with $b_n = (p+1)(p^{n-1}-1)$ for $n > 1$. Now, if $s \geq 1$ and $p^i | j \leq a_{n-i}$ with $j \leq p^n$ if $s = 1$, then $x_n^s \in Ext^{0,*}_{BP_*BP}(BP_*, BP_*/(p^{i+1}, v_1^j))$. Define

$$\beta_{sp^n/j,i+1} := \delta' \delta''(x_n^s) \in Ext^{2,*}_{BP_*BP}(BP_*, BP_*) \tag{4.16}$$

where $\delta'$ (resp. $\delta''$) is the boundary-homomorphism associated to $E'$ (resp. $E''$)

$$E' : 0 \to \Omega(BP_*) \xrightarrow{p^{i+1}} \Omega(BP_*) \to \Omega(BP_*/p^{i+1}) \to 0, \tag{4.17}$$

$$E'' : 0 \to \Omega(BP_*/p^{i+1}) \xrightarrow{v_1^j} \Omega(BP_*/p^{i+1}) \to \Omega(BP_*/(p^{i+1}, v_1^j)) \to 0, \tag{4.18}$$

where we let $\Omega(-)$ denote $\Omega_{BP_*BP}(-)$. We often abbreviate $\beta_{sp^n/j,1}$ to $\beta_{sp^n/j}$ and $\beta_{sp^n/1}$ to $\beta_{sp^n}$. When we work with $\beta$-family elements in practice, we require the indexes $(s, n, j, i)$ to satisfy certain relations as specified in the following theorem.

Theorem 4.8 ([11, 12]). Let $p$ be an odd prime. $Ext^{2,*}_{BP_*BP}(BP_*, BP_*)$ is the direct sum of cyclic subgroups generated by $\beta_{sp^n/j,i+1}$ for $n \geq 0$, $p \nmid s \geq 1$, $j \geq 1$, $i \geq 0$, subject to: (1) $j \leq p^n$, if $s = 1$, (2) $p^i | j \leq a_{n-i}$, and (3) $a_{n-i-1} < j$, if $p^{i+1} | j$.

Proposition 4.9. Concerning the image of the $\beta$-family elements under the map $\phi$ specified in (4.1), we have

(1) $\phi(\beta_1) = -e_{4,1}$.
(2) $\phi(\beta_2) = 2k_0$.
(3) $\phi(\beta_s) = 0$, for $s > 2$.
(4) $\phi(\beta_{p^n/p^n}) = -e_{4,n+1}$, for $n \geq 1$.
(5) $\phi(\beta_{sp^n/p^n}) = 0$, for $n \geq 1$, $s \geq 2$.

Proof. (1) and (2) are computed in [5, Lemma 3.4]. Note the elements $b_0 := h_{1,1}h_{3,2} + h_{2,1}h_{2,0} + h_{3,1}h_{1,1}$, $k_0 := h_{2,0}h_{1,1}$ defined in [5, Theorem 2.7] correspond to $e_{4,1}$ and $k_0$ respectively in our notation, see Proposition 3.3.

Before proving (3), (4), (5), we first introduce some notations.

Consider $\beta_{sp^n/p^n} = \delta'\delta''(x_n^s)$, we denote

$$y_{sp^n/p^n} := \delta''(x_n^s) = \frac{d'(x_n^s)}{v_1^{p^n}} \in \Omega(BP_*/p), \tag{4.19}$$

where we let $x_n^s$ also denote the preimage of $x_n^s$ with respect to the map $\Omega(BP_*/p) \to \Omega(BP_*/(p, v_1^{p^n}))$ and let $d'$ denote the differential map of the cobar complex $\Omega(BP_*/p)$.

Similarly, using the definition of the connecting homomorphism $\delta'$, we have

$$\beta_{sp^n/p^n} = \delta'(y_{sp^n/p^n}) = \frac{d(y_{sp^n/p^n})}{p} \in \Omega(BP_*), \tag{4.20}$$

where we let $y_{sp^n/p^n}$ also denote the preimage of $y_{sp^n/p^n}$ with respect to the map $\Omega(BP_*) \to \Omega(BP_*/p)$ and let $d$ denote the differential map of the cobar complex $\Omega(BP_*)$.

(3) Let $n = 0$, $s \geq 3$. Then we have:

$$y_s = \delta''(v_2^s) = \frac{d'(v_2^s)}{v_1} \in I[s-1] \cdot \Omega(BP_*/p). \tag{4.21}$$

$$\beta_s = \delta'(y_s) = \frac{d(y_s)}{p} \in I[s-2] \cdot \Omega(BP_*) \subset I_3 \cdot \Omega(BP_*). \tag{4.22}$$

Upon reduction modulo $I_3$, we have $\eta(\beta_s) = 0$. Then $\phi(\beta_s) = 0$ for $s \geq 3$.

(4) Let $n \geq 1$. We claim that in $\Omega(BP_*/(p, v_1^{p^n}))$, we can express $x_n$ as $v_2^{p^n} + L_n$, where $L_n \in I[2p^n - p^{n-1}] \cdot \Omega(BP_*/(p, v_1^{p^n}))$.

If $n = 1$, we have $x_1 = v_2^p - v_1^p v_2^{-1} v_3 = v_2^p \in \Omega(BP_*/(p, v_1^p))$ since $v_1^p = 0 \in \Omega(BP_*/(p, v_1^p))$. If $n = 2$, we have $x_2 = x_1^p - v_1^{p^2-1} v_2^{p^2-p+1} - v_1^{p^2+p-1} v_2^{p^2-2p} v_3 = v_2^{p^2} - v_1^{p^2-1} v_2^{p^2-p+1} \in \Omega(BP_*/(p, v_1^{p^2}))$ since $p, v_1^{p^2} = 0 \in \Omega(BP_*/(p, v_1^{p^2}))$. The case for general $n \geq 3$ can be proved analogously.

Consequently, we have

$$y_{p^n/p^n} = \delta''(x_n) = \frac{d'(x_n)}{v_1^{p^n}} = \frac{d'(v_2^{p^n})}{v_1^{p^n}} + \frac{d'(L_n)}{v_1^{p^n}}. \tag{4.23}$$

$$\beta_{p^n/p^n} = \delta'(y_{p^n/p^n}) = \frac{d(y_{p^n/p^n})}{p} = \frac{1}{p} d\left(\frac{d'(v_2^{p^n})}{v_1^{p^n}}\right) + \frac{1}{p} d\left(\frac{d'(L_n)}{v_1^{p^n}}\right). \tag{4.24}$$

Note that $L_n \in I[2p^n - p^{n-1}] \cdot \Omega(BP_*/(p, v_1^{p^n}))$. Analogous to Lemma 4.3, we have

$$\frac{d'(L_n)}{v_1^{p^n}} \in I[p^n - p^{n-1}] \cdot \Omega(BP_*/p), \tag{4.25}$$

$$\frac{1}{p} d\left(\frac{d'(L_n)}{v_1^{p^n}}\right) \in I[p^n - p^{n-1} - 1] \cdot \Omega(BP_*) \subset I_3 \cdot \Omega(BP_*). \tag{4.26}$$

Using (2.3), we can write $\eta_R(v_2) = v_2 + v_1 t_1^p + pt_2 + L$, where $L \in (p^2, v_1^p) \cdot BP_*BP$. Since $p = 0$ in $\Omega(BP_*/p)$, we can write:

$$\frac{d'(v_2^{p^n})}{v_1^{p^n}} = -t_1^{p^{n+1}} + L_{p^n}, \tag{4.27}$$

where $L_{p^n} \in I[p^{n+1} - p^n] \cdot \Omega(BP_*/p)$. This implies

$$\frac{1}{p} d(L_{p^n}) \in I[p^{n+1} - p^n - 1] \cdot \Omega(BP_*) \subset I_3 \cdot \Omega(BP_*). \tag{4.28}$$

Moreover, we have

$$\begin{aligned} \frac{1}{p} d(-t_1^{p^{n+1}}) &= -\frac{1}{p}[\Delta(t_1^{p^{n+1}}) - 1 \otimes t_1^{p^{n+1}} - t_1^{p^{n+1}} \otimes 1] \\ &= -\frac{1}{p}[(1 \otimes t_1 + t_1 \otimes 1)^{p^{n+1}} - 1 \otimes t_1^{p^{n+1}} - t_1^{p^{n+1}} \otimes 1] \\ &= -b_{1,n}, \end{aligned} \tag{4.29}$$

as defined in Notations 2.3.

Combining (4.24) ~ (4.29), we have $\eta(\beta_{p^n/p^n}) = -b_{1,n}$ using Notations 2.6.

Next, we will show that $\psi(b_{1,n}) = e_{4,n+1} \in H^{*,*}S(3)$ for $n \geq 0$. Then, we can conclude that $\phi(\beta_{p^n/p^n}) = -e_{4,n+1}$, for $n \geq 1$.

Following Notations 2.6, we have $\psi(b_{1,n}) = \tilde{b}_{1,n}$. By Proposition 2.5, in the cobar complex $\Omega^{*,*}_{S(3)}(\mathbb{F}_p)$, we have $d(t_4) = t_1 \otimes t_3^p + t_2 \otimes t_2^{p^2} + t_3 \otimes t_1^{p^3} - \tilde{b}_{1,2}$. Hence, we have equivalent cohomology classes $[\tilde{b}_{1,2}] = [t_1 \otimes t_3^p + t_2 \otimes t_2^{p^2} + t_3 \otimes t_1^{p^3}] = e_{4,3}$. This implies $\psi(b_{1,2}) = e_{4,3}$.

Note that if $a$ is not a multiple of $p$, then $a^p \equiv a$ modulo $p$. Hence, working over $\mathbb{F}_p$, we have $\tilde{b}_{1,n+1} = \tilde{b}_{1,n}^p$. Moreover, note that $t_1^{p^3} = t_1$ in $S(3)$, so we have $\tilde{b}_{1,n+3} = \tilde{b}_{1,n}$. Similarly, one can show that $e_{4,n+1} = e_{4,n}^p$ and $e_{4,n+3} = e_{4,n}$. Hence, we conclude that $\psi(b_{1,n}) = e_{4,n+1}$ for each $n \geq 0$.

This finishes the proof for statement (4).

(5) Let $n \geq 1$, $s \geq 2$. Direct observation shows $x_n \in I[p^n - p^{n-1}] \cdot \Omega(BP_*/(p, v_1^{p^n}))$. From this, we can conclude

$$x_n^s \in I[sp^n - sp^{n-1}] \cdot \Omega(BP_*/(p, v_1^{p^n})), \tag{4.30}$$

$$y_{sp^n/p^n} \in I[sp^n - sp^{n-1} - p^n] \cdot \Omega(BP_*/p), \tag{4.31}$$

$$\beta_{sp^n/p^n} \in I[sp^n - sp^{n-1} - p^n - 1] \cdot \Omega(BP_*). \tag{4.32}$$

Note $sp^n - sp^{n-1} - p^n - 1 = (sp - s - p)p^{n-1} - 1 \geq sp - s - p - 1 \geq p - 3 > 1$. We have $\eta(\beta_{sp^n/p^n}) = 0$. Then $\phi(\beta_{sp^n/p^n}) = 0$, for $n \geq 1$, $s \geq 2$.

□

4.10. $\gamma$-family elements. Analogous to $\alpha$-family and $\beta$-family elements, one can construct $\gamma$-family elements in $Ext^{3,*}_{BP_*BP}(BP_*, BP_*)$. For the purpose of this paper, we only need to consider $\gamma_s$ for $s \geq 1$. Recall the following result concerning $\phi(\gamma_s)$ (see [5, Lemma 3.4], also [3, Lemma 4.1]).

Proposition 4.11. Concerning the image of the $\gamma$-family elements under the map $\phi$ specified in (4.1), we have

$$\phi(\gamma_s) = -s(s^2 - 1)v_0 + s(s-1)\rho k_1, \quad s \geq 1.$$

Remark 4.12. Here, the result in Proposition 4.11 differs from the formula in [3, 5] by a negative sign, as our definitions of the differential in the cobar complex (Definition 2.9) differ by a negative sign.

4.13. $\zeta$-family elements. Let $p \geq 7$, $s \geq 1$. It is proved in [11, 15, 19, 20] that $\alpha_s$, $\beta_s$, $\gamma_s$ all represent nontrivial elements in $\pi_*(S)$. Using the Adams spectral sequence, Cohen [2] also found another family of nontrivial elements $\zeta_n \in \pi_*(S)$, for $n \geq 1$. We also denote the representative of $\zeta_n$ in the Adams-Novikov $E_2$-page $Ext^{*,*}_{BP_*BP}(BP_*, BP_*)$ by $\zeta_n$.

Cohen [2] shows $\zeta_n = \alpha_1\beta_{p^n/p^n} + \alpha_1 x \in Ext^{3,*}_{BP_*BP}(BP_*, BP_*)$, where $x = \sum\limits_{s,k,j} a_{s,k,j}\beta_{sp^k/j}$, $0 \leq a_{s,k,j} \leq p-1$, and $a_{1,n,p^n} = 0$. Moreover, by comparing the inner degrees, one can show [3] $x = \sum a_{s,k,p^k}\beta_{sp^k/p^k}$, where $k \leq n$, $s = \frac{p^{n-k+1}+1}{p+1} \neq 1$. Then, simple calculation shows that $s > 2$.

Proposition 4.14. Concerning the image of the $\zeta$-family elements under the map $\phi$ specified in (4.1), we have

$$\phi(\zeta_n) = h_{1,0}e_{4,n+1}, \quad n \geq 1.$$

Proof. We have $\zeta_n = \alpha_1\beta_{p^n/p^n} + \alpha_1 x$. By Proposition 4.4 and Proposition 4.9, we have $\phi(\alpha_1) = -h_{1,0}$, $\phi(\beta_{p^n/p^n}) = -e_{4,n+1}$, and $\phi(x) = 0$. Therefore, $\phi(\zeta_n) = \phi(\alpha_1)\phi(\beta_{p^n/p^n}) = h_{1,0}e_{4,n+1}$. □

Gathering the analysis of the $\alpha$-family, $\beta$-family, $\gamma$-family, and $\zeta$-family elements, we have the following result.

Proposition 4.15. Under the comparison map $\phi : Ext^{*,*}_{BP_*BP}(BP_*, BP_*) \to H^{*,*}(S(3))$, all nonzero images of $\{\alpha_s, \beta_s, \gamma_s, \zeta_s | s \geq 1\}$ are listed as follows:
(1) $\phi(\alpha_1) = -h_{1,0}$,
(2) $\phi(\beta_1) = -e_{4,1}$
(3) $\phi(\beta_2) = 2k_0$,
(4) $\phi(\gamma_s) = -s(s^2-1)v_0 + s(s-1)\rho k_1$, for $s \not\equiv 0, 1$ mod p.
(5) $\phi(\zeta_n) = h_{1,0}e_{4,n+1}$, for $n \geq 1$.

Proof. This follows directly from Propositions 4.4, 4.9, 4.11, and 4.14. Note $h_{1,0}e_{4,0} = e_{4,0}h_{1,0} \neq 0$ since $e_{4,0}h_{1,0}$ is a generator of $H^{3,*}(S(3))$ (Proposition 3.3). Similarly, $h_{1,0}e_{4,2} = e_{4,2}h_{1,0} \neq 0$. Finally, $h_{1,0}e_{4,1} \neq 0$ since $e_{4,1}e_{4,2}h_{1,0} \neq 0$ is a generator in dimension 5. Therefore, $\phi(\zeta_n) \neq 0$ for $n \geq 1$. □

## 5. Detection of nontrivial products in $\pi_*(S)$

In this section, we prove Theorems 1.1 and 1.3.

Proof of Theorem 1.1. We consider the representation of $\beta_1\gamma_s\zeta_n$ on the $E_2$-page of the ANSS. According to Propositions 4.15 and 3.6, we have

$$\begin{aligned}
\phi(\beta_1\gamma_s\zeta_n) &= e_{4,1}(s(s^2-1)v_0 - s(s-1)\rho k_1)h_{1,0}e_{4,n+1} \\
&= s(s^2-1)e_{4,1}v_0h_{1,0}e_{4,n+1} - s(s-1)\rho e_{4,1}k_1h_{1,0}e_{4,n+1} \\
&= s(s^2-1)e_{4,1}v_0h_{1,0}e_{4,0} \qquad (n \equiv 2 \text{ mod } 3,\ h_{1,0}k_1 = 0) \\
&= \frac{s(s^2-1)}{3}e_{4,1}^2e_{4,0}g_2 \\
&\neq 0
\end{aligned}$$

when $n \equiv 2 \bmod 3$, and $s \not\equiv 0, \pm 1 \bmod p$.

Hence, we conclude $\beta_1\gamma_s\zeta_n \neq 0 \in Ext^{8,*}_{BP_*BP}(BP_*, BP_*)$. Since $\beta_1$, $\gamma_s$, and $\zeta_n$ are all permanent cycles in the ANSS, their product is also a permanent cycle.

Note the differentials of the ANSS have the form $d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}$, where $r \geq 2$. Additionally, the inner degrees of the elements in the ANSS are all multiples of $q = 2p - 2$. Thus, the first potentially nontrivial differentials in the ANSS occur at $d_{2p-1}$. Suppose $\beta_1\gamma_s\zeta_n$ is in the target of a differential $d_r$. Then we have $8 = s + r \geq 2p - 1 \geq 13$. This is a contradiction. Hence, $\beta_1\gamma_s\zeta_n$ is not in the target of any differential in the ANSS. This proves $\beta_1\gamma_s\zeta_n$ survives to the nontrivial product $\beta_1\gamma_s\zeta_n \neq 0 \in \pi_*(S)$. □

Proof of Theorem 1.3. Let $X = X_1X_2 \cdots X_m$ be a product in $\pi_*(S)$ where each factor belongs to the set $\{\alpha_s, \beta_s, \gamma_s, \zeta_s | s \geq 1\}$. Let $x = x_1x_2 \cdots x_m \in Ext^{*,*}_{BP_*BP}(BP_*, BP_*)$ represent $X$ on the Adams-Novikov $E_2$-page. If $X$ can be detected as nontrivial by comparing with $H^{*,*}S(3)$, then we have $\phi(x) \neq 0 \in H^{*,*}(S(3))$.

On the other hand, if $0 \neq \phi(x) \in H^{a,*}(S(3))$, then it follows that $a \leq 9$. Similar to the arguments in the proof of Theorem 1.1, we can show that $x$ can not be in the target of any differential in the ANSS by degree reasons. Hence, the product $X$ is nontrivial in $\pi_*(S)$.

For the rest of the proof, our task is to find all products $X$ such that $\phi(x) \neq 0$. Note $\phi(x) \neq 0$ implies $\phi(x_i) \neq 0$ for $1 \leq i \leq m$. Then $X_i \in \{\alpha_1, \beta_1, \beta_2, \gamma_s, \zeta_n | s \not\equiv 0, 1 \bmod p, n \geq 1\}$ by Proposition 4.15.

We first consider binary products. By Propositions 3.3, 3.6, and 4.15, we have:

(1) $\phi(\alpha_1\alpha_1) = h_{1,0}^2 = 0$,
(2) $\phi(\alpha_1\beta_1) = h_{1,0}e_{4,1} \neq 0$.
(3) $\phi(\alpha_1\beta_2) = -2h_{1,0}k_0 \neq 0$, by Example 3.5.
(4) $\phi(\alpha_1\gamma_s) = s(s^2-1)h_{1,0}v_0 + s(s-1)\rho h_{1,0}k_1 = s(s^2-1)h_{1,0}v_0 \neq 0$, if and only if $s \not\equiv 0, \pm 1$ mod p, since $h_{1,0}v_0 \neq 0$, $h_{1,0}k_1 = 0$.
(5) $\phi(\alpha_1\zeta_n) = -h_{1,0}^2 e_{4,n+1} = 0$,
(6) $\phi(\beta_1^2) = e_{4,1}^2 \neq 0$, since $e_{4,1}^2 \in H^{*,*}(S(3))$ is a generator.
(7) $\phi(\beta_1\beta_2) = -2e_{4,1}k_0 = 0$, by Example 3.5.
(8) $\phi(\beta_1\gamma_s) = s(s^2-1)e_{4,1}v_0 - s(s-1)\rho e_{4,1}k_1 \neq 0$, if and only if $s \not\equiv 0, 1$ mod p by Proposition 3.6.
(9) $\phi(\beta_1\zeta_n) = -e_{4,1}h_{1,0}e_{4,n+1} \neq 0$, since $h_{1,0}e_{4,1}^2$, $h_{1,0}e_{4,1}e_{4,2}$, and $h_{1,0}e_{4,1}e_{4,0} = e_{4,0}^2h_{1,1}$ are all generators in $H^{*,*}(S(3))$.
(10) $\phi(\beta_2^2) = 4k_0^2 = 0$, by Example 3.5.
(11) $\phi(\beta_2\gamma_s) = -2s(s^2-1)k_0v_0 + 2s(s-1)\rho k_0k_1 = -2s(s^2-1)k_0v_0 \neq 0$, if and only if $s \not\equiv 0, \pm 1$ mod p,
(12) $\phi(\beta_2\zeta_n) = 2k_0h_{1,0}e_{4,n+1} = 0$,
(13) $\phi(\gamma_s\gamma_t) = 0$, since $\rho^2 = 0$, $v_0^2 = 0$, and $v_0k_1 = 0$ by direct computation similar to Example 3.5.
(14) $\phi(\gamma_s\zeta_n) = -s(s^2-1)v_0h_{1,0}e_{4,n+1} + s(s-1)\rho k_1h_{1,0}e_{4,n+1} = -s(s^2-1)v_0h_{1,0}e_{4,n+1} \neq 0$, if and only if $n \not\equiv 1 \bmod 3$, $s \not\equiv 0, \pm 1 \bmod p$. Note $v_0h_{1,0}e_{4,2} = 0$ by direct computation similar to Example 3.5. Besides, we note that $v_0h_{1,0}e_{4,0} \neq 0$ and $v_0h_{1,0}e_{4,1} \neq 0$. These assertions follow from the result that $e_{4,1}v_0h_{1,0}e_{4,0} \neq 0$ by Proposition 3.6.
(15) $\phi(\zeta_m\zeta_n) = 0$, since $h_{1,0}^2 = 0$.

For triple products, if $X = X_1X_2X_3 \neq 0$, then $X_1X_2$, $X_2X_3$, and $X_1X_3$ are all nontrivial. By the above analysis, we only need to consider the following products.

(1) $\phi(\alpha_1\beta_1^2) = -h_{1,0}e_{4,1}^2 \neq 0$.
(2) $\phi(\alpha_1\beta_1\gamma_s) = -s(s^2-1)h_{1,0}e_{4,1}v_0 - s(s-1)\rho h_{1,0}e_{4,1}k_1 = -s(s^2-1)h_{1,0}e_{4,1}v_0 \neq 0$, if and only if $s \not\equiv 0, \pm 1$ mod p. Note $h_{1,0}e_{4,1}v_0 \neq 0$ since $e_{4,1}v_0h_{1,0}e_{4,0} \neq 0$ by Proposition 3.6.
(3) $\phi(\alpha_1\beta_2\gamma_s) = 2s(s^2-1)h_{1,0}k_0v_0 + 2s(s-1)\rho h_{1,0}k_0k_1 = 0$, since $h_{1,0}k_0v_0, h_{1,0}k_1 = 0$ by Proposition 3.6.
(4) $\phi(\beta_1^3) = -e_{4,1}^3 = 0$ by direct computation similar to Example 3.5.
(5) $\phi(\beta_1^2\gamma_s) = -s(s^2-1)e_{4,1}^2v_0 + s(s-1)\rho e_{4,1}^2k_1 = 0$, since $e_{4,1}^2v_0, e_{4,1}^2k_1 = 0$ by direct computation similar to Example 3.5.
(6) $\phi(\beta_1^2\zeta_n) = h_{1,0}e_{4,1}^2e_{4,n+1} \neq 0$, if and only if $n \equiv 1$ mod 3. Note $h_{1,0}e_{4,1}^2e_{4,0}$, $h_{1,0}e_{4,1}^3 = 0$ by direct computation. Using the formula $e_{4,i+1}h_{1,i} = e_{4,i}h_{1,i+1}$ from Proposition 3.6, we have $h_{1,0}e_{4,1}^2e_{4,2} = e_{4,1}^2e_{4,2}h_{1,0} = e_{4,1}^2e_{4,0}h_{1,2} \neq 0$ since $e_{4,1}^2e_{4,0}h_{2,2}h_{1,2} = e_{4,1}^2e_{4,0}g_2 \neq 0$ is a generator.
(7) $\phi(\beta_1\gamma_s\zeta_n) = s(s^2-1)e_{4,1}v_0h_{1,0}e_{4,n+1} \neq 0$, if and only if $n \equiv 2$ mod 3, $s \not\equiv 0, \pm 1$ mod p. Note we have $e_{4,1}v_0h_{1,0}e_{4,0} \neq 0$ by Proposition 3.6. On the other hand, $e_{4,1}v_0h_{1,0}e_{4,1} = 0$, and $e_{4,1}v_0h_{1,0}e_{4,2} = 0$ by direct computation similar to Example 3.5.

For four-fold products, if $X = X_1X_2X_3X_4 \neq 0$, then $X_1X_2X_3$, $X_2X_3X_4$, $X_1X_3X_4$, and $X_1X_2X_4$ are all nontrivial. By the above analysis, there are no nontrivial four-fold products.

In this way, we have found all nontrivial products of the desired form. The result is summarized as Theorem 1.3.

□

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