DETECTING NONTRIVIAL PRODUCTS IN THE STABLE HOMOTOPY RING OF SPHERES VIA THE THIRD MORAVA STABILIZER ALGEBRA

Abstract. Let $p \geq 7$ be a prime number. Let $S(3)$ denote the third Morava stabilizer algebra. In recent years, Kato-Shimomura and Gu-Wang-Wu found several families of nontrivial products in the stable homotopy ring of spheres $\pi_*(S)$ using $H^{*,*}(S(3))$. In this paper, we determine all nontrivial products in $\pi_*(S)$ of the Greek letter family elements $\alpha_s, \beta_s, \gamma_s$ and Cohen's elements ζ_n which are detectable by $H^{*,*}(S(3))$. In particular, we show $\zeta_n\beta_1\gamma_s \neq 0 \in \pi_*(S)$, if $n \equiv 2 \bmod 3$, $s \not\equiv 0, \pm 1 \bmod p$.

1. Introduction

The computation of the ring of stable homotopy groups of spheres, denoted as $\pi_*(S)$, is one of the fundamental problems in algebraic topology. The Adams-Novikov spectral sequence (ANSS) based on the Brown-Peterson spectrum BP is an incredibly powerful tool for computing the p-component of $\pi_*(S)$, where p is a prime number. The E_2-page of the ANSS is of the form $Ext^{s,t}_{BP_*BP}(BP_*, BP_*)$ and has been extensively studied in low dimensions.

For $s = 1$, $Ext^{1,*}_{BP_*BP}(BP_*, BP_*)$ is generated by $\alpha_{kp^n/n+1}$ for $n \geqslant 0$, and $p \nmid k \geqslant 1$ ([14]).

For $s = 2$, $Ext^{2,*}_{BP_*BP}(BP_*, BP_*)$ is generated by $\beta_{kp^n/j,i+1}$ for suitable (n, k, j, i) ([10, 11]).

For $s = 3$, only partial results of $Ext^{3,*}_{BP_*BP}(BP_*, BP_*)$ are known (see, for example, [12, 13, 17]). Nonetheless, a construction of a family of linearly independent elements denoted as $\gamma_{s_3/s_2,s_1}$ in $Ext^{3,*}_{BP_*BP}(BP_*, BP_*)$ has been achieved ([10]).

Through the computations of $Ext^{s,t}_{BP_*BP}(BP_*, BP_*)$ in low dimensions, numerous nontrivial elements in $\pi_*(S)$ can be obtained. In particular, for $p \geq 7$, there are the Greek letter family elements, denoted as α_s, β_s, and γ_s with $s \geq 1$ [10, 14, 18, 19]. These families are represented by elements of the same name in $Ext^{1,*}_{BP_*BP}(BP_*, BP_*)$, $Ext^{2,*}_{BP_*BP}(BP_*, BP_*)$, and $Ext^{3,*}_{BP_*BP}(BP_*, BP_*)$, respectively.

Furthermore, using the Adams spectral sequence, Cohen [2] discovered another family of nontrivial elements $\zeta_n \in \pi_*(S)$ with $n \geq 1$. The representation of ζ_n in $Ext^{3,*}_{BP_*BP}(BP_*, BP_*)$ has also been studied in [2] (also see [3]).

Nontrivial products on $\pi_*(S)$. There exists a natural ring structure on $\pi_*(S)$ in which multiplication is defined by the composition of representing maps. In order to gain a deeper understanding of the ring structure of $\pi_*(S)$, it is necessary to determine whether the product of certain given elements is trivial. The main purpose of this paper is to find nontrivial products formed by the elements in $\{\alpha_s, \beta_s, \gamma_s, \zeta_s | s \geq 1\}$. To ensure the well-definedness of these elements, we assume $p \geq 7$ for the remainder of the paper, unless otherwise specified.

Numerous results have been obtained in this direction. Just to mention a few:

(a) Aubry [1] shows that $\alpha_1\beta_2\gamma_2, \beta_1^r\beta_2\gamma_2 \neq 0$ if $r \leq p-1$.

(b) Lee-Ravenel [6] shows $\beta_1^{p^2-p-1} \neq 0$ for $p \geq 7$.

2020 *Mathematics Subject Classification.* 55Q45, 55Q51, 55T15.

Keywords: Stable homotopy of spheres, v_n-periodicity, Adams spectral sequences.

(c) Lee [7] shows: (1) $\beta_1^r\beta_s, \beta_1^{r-1}\beta_2\beta_{kp-1} \neq 0$ for $p \geq 5$, if $r, k \leq p-1$, $s < p^2 - p - 1$, and $s \not\equiv 0$ mod p; (2) $\beta_1^r\gamma_t, \beta_1^{r-1}\beta_2\gamma_t \neq 0$, if $r, t \leq p-1$; (3) $\alpha_1\beta_1^r\gamma_t \neq 0$, if $r \leq p-2$, $2 \leq t \leq p-1$; (4) $\beta_1^{p-1}\zeta_n \neq 0$.
(d) Liu-Liu [8] shows that $\alpha_1\beta_1^2\beta_2\gamma_s \neq 0$ if $4 < s < p$.
(e) Zhao-Wang-Zhong [22] shows that $\gamma_{p-1}\zeta_n \neq 0$ if $n \neq 4$.

In recent years, Kato-Shimomura [5] have developed a method for detecting nontrivial products on $\pi_*(S)$ through the use of $S(3)$, where $S(3)$ denotes the third Morava stabilizer algebra [15]. This new approach offers an advantage when studying products involving γ_s for arbitrarily large values of s. We can briefly recall their strategy as follows.

There exists a natural map $\phi : Ext^{*,*}_{BP_*BP}(BP_*, BP_*) \to Ext^{*,*}_{S(3)}(\mathbb{F}_p, \mathbb{F}_p) =: H^{*,*}(S(3))$. The cohomology $H^{*,*}(S(3))$ is computed in [3, 17, 21]. Given a product $x = x_1x_2\cdots x_n \in \pi_*(S)$, we let $y = y_1y_2\cdots y_n \in Ext^{*,*}_{BP_*BP}(BP_*, BP_*)$ represent x on the E_2-page of the ANSS. If $\phi(y) \neq 0$, then $y \neq 0 \in Ext^{*,*}_{BP_*BP}(BP_*, BP_*)$. For the examples of interest, y will not be eliminated by any Adams-Novikov differential due to degree considerations. Thus, we can conclude that $x \neq 0 \in \pi_*(S)$ in this case.

Using this strategy, Kato-Shimomura [5] demonstrate the following: (1) $\alpha_1\gamma_s \neq 0$, if $s \not\equiv 0, \pm1$ mod p; (2) $\beta_1\gamma_s \neq 0$, if $s \not\equiv 0, 1$ mod p; (3) $\beta_2\gamma_s \neq 0$, if $s \not\equiv 0, \pm1$ mod p.

Similarly, Gu-Wang-Wu [3] show that $\zeta_n\gamma_s \neq 0$ if $n \not\equiv 1$ mod 3 and $s \not\equiv 0, \pm1$ mod p.

Our main results. In this paper, we employ the "Detection via $H^{*,*}(S(3))$" method, which was developed in [3, 5], to detect nontrivial products on $\pi_*(S)$. However, instead of focusing on specific examples, we fully utilize the potential of this method and enumerate all detectable products. The main results of our study are as follows:

Theorem 1.1. *Let $p \geq 7$ be a prime. Let $n \equiv 2$ mod 3, and $s \not\equiv 0, \pm1$ mod p. Then $\zeta_n\beta_1\gamma_s \neq 0 \in \pi_*(S)$.*

Theorem 1.2. *Let $p \geq 7$ be a prime. We consider the products in $\pi_*(S)$ where each factor belongs to $\{\alpha_s, \beta_s, \gamma_s, \zeta_s : s \geq 1\}$. Among all such products, only the following ones can be detected as nontrivial products using the comparison with $H^{*,*}S(3)$.*

i) $\alpha_1\beta_1$,
ii) $\alpha_1\beta_2$,
iii) $\alpha_1\gamma_s$, *if* $s \not\equiv 0, \pm1$ *mod p,*
iv) $\beta_1\gamma_s$, *if* $s \not\equiv 0, 1$ *mod p,*
v) $\beta_2\gamma_s$, *if* $s \not\equiv 0, \pm1$ *mod p,*
vi) $\alpha_1\beta_1\gamma_s$, *if* $s \not\equiv 0, \pm1$ *mod p,*
vii) $\zeta_n\gamma_s$, *if* $n \not\equiv 1$ *mod 3,* $s \not\equiv 0, \pm1$ *mod p,*
viii) $\zeta_n\beta_1$, *if* $n \not\equiv 0$ *mod 3,*
ix) $\zeta_n\beta_1\gamma_s$, *if* $n \equiv 2$ *mod 3,* $s \not\equiv 0, \pm1$ *mod p.*

The non-triviality of i) ∼ viii) has been determined by earlier works in [3, 5, 7, 10]. We single out the new result *ix*) as Theorem 1.1. We have exhausted the potential of the "Detection via $H^{*,*}(S(3))$" strategy in Theorem 1.2. To detect other nontrivial products in $\pi_*(S)$, different methods would need to be employed.

Organization of the paper. In Section 2, we review the basic structures of the Hopf algebroid (BP_*, BP_*BP) and the third Morava stabilizer algebra $S(3)$. In Section 3, we review the $\mathbb{F}_p$-algebra structure of $H^{*,*}S(3)$. In Section 4, we recall the constructions of the α, β, γ-family elements in the Adams-Novikov spectral sequence. Then we determine their images under the comparison map $\phi : Ext^{*,*}_{BP_*BP}(BP_*, BP_*) \to H^{*,*}(S(3))$. In Section 5, we use the

$\mathbb{F}_p$-algebra structure of $H^{*,*}S(3)$ to detect nontrivial products in $Ext^{*,*}_{BP_*BP}(BP_*, BP_*)$. Then we prove Theorem 1.1 and Theorem 1.2.

2. Hopf algebroids

This section recalls the basic definitions and constructions related to Hopf algebroids. In particular, we review the basic structures of the Hopf algebroid (BP_*, BP_*BP) and the third Morava stabilizer algebra $S(3)$.

2.1. The Hopf algebroid (BP_*, BP_*BP).

Definition 2.2. A *Hopf algebroid* over a commutative ring K is a pair (A, Γ) of commutative K-algebras with structure maps

$$\begin{gathered}\text{left unit map } \eta_L : A \to \Gamma \\ \text{right unit map } \eta_R : A \to \Gamma \\ \text{coproduct map } \Delta : \Gamma \to \Gamma \otimes_A \Gamma \\ \text{counit map } \varepsilon : \Gamma \to A \\ \text{conjugation map } c : \Gamma \to \Gamma\end{gathered}$$

such that for any other commutative K-algebra B, the two sets $\mathrm{Hom}(A, B)$ and $\mathrm{Hom}(\Gamma, B)$ are the objects and morphisms of a groupoid.

An important example of Hopf algebroids is (BP_*, BP_*BP). Recall that we have

$$BP_* := \pi_*(BP) = \mathbb{Z}_{(p)}[v_1, v_2, \cdots], \quad BP_*BP = BP_*[t_1, t_2, \cdots] \tag{2.1}$$

where the inner degrees are $|v_n| = |t_n| = 2(p^n - 1)$. Throughout this paper, we denote $v_0 = p$, and $t_0 = 1$. The structure maps of the Hopf algebroid (BP_*, BP_*BP) are described in [4, 10, 17]. In practice, the following formulas [5] are useful.

$$\eta_R(v_1) = v_1 + pt_1, \tag{2.2}$$

$$\eta_R(v_2) \equiv v_2 + v_1 t_1^p + pt_2 \mod (p^2, v_1^p), \tag{2.3}$$

$$\Delta(t_1) = t_1 \otimes 1 + 1 \otimes t_1, \tag{2.4}$$

$$\Delta(t_2) = t_2 \otimes 1 + t_1 \otimes t_1^p + 1 \otimes t_2 - v_1 b_{1,0}. \tag{2.5}$$

Notations 2.3. We denote $b_{i,j} = \frac{1}{p}[(\sum_{k=0}^{i} t_{i-k} \otimes t_k^{p^{i-k}})^{p^{j+1}} - \sum_{k=0}^{i} t_{i-k}^{p^{j+1}} \otimes t_k^{p^{i-k+j+1}}]$ for $i \geq 1$, $j \geq 0$. See [20] for related discussions.

2.4. **Morava stabilizer algebras.** We recall the basic properties of the Morava stabilizer algebras, which are studied in detail in [9, 15].

Let $K(n)_*$ denote $\mathbb{F}_p[v_n, v_n^{-1}]$. We can equip $K(n)_*$ with a BP_*-algebra structure via the ring homomorphism which sends all v_i with $i \neq n$ to 0. Then we define $\Sigma(n) := K(n)_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} K(n)_*$. As an algebra, one has $\Sigma(n) \cong K(n)_*[t_1, t_2, \cdots]/(v_n t_i^{p^n} - v_n^{p^i} t_i | i > 0)$. The coproduct structure of $\Sigma(n)$ is inherited from that of BP_*BP.

Moreover, one can prove $Ext^{*,*}_{BP_*BP}(BP_*, v_n^{-1}BP_*/I_n) \cong Ext^{*,*}_{\Sigma(n)}(K(n)_*, K(n)_*)$, where we let I_n denote the ideal $(p, v_1, v_2, \cdots, v_{n-1}) \subset BP_*$.

We define the Hopf algebra $S(n) := \Sigma(n) \otimes_{K(n)_*} \mathbb{F}_p$, where $K(n)_*$ and $\Sigma(n)$ are here regarded as graded over $\mathbb{Z}/2(p^n - 1)$ and $\mathbb{F}_p$ is a $K(n)_*$-algebra via the map sending v_n to 1. We call $S(n)$ the n-th Morava stabilizer algebra. One can show

$$Ext^{*,*}_{\Sigma(n)}(K(n)_*, K(n)_*) \otimes_{K(n)_*} \mathbb{F}_p \cong Ext^{*,*}_{S(n)}(\mathbb{F}_p, \mathbb{F}_p) =: H^{*,*}(S(n)) \tag{2.6}$$

For the purpose of this paper, from now on, we will only consider the case when $n = 3$. We have the following results.

Proposition 2.5 ([16])**.** *As an algebra,* $S(3) \cong \mathbb{F}_p[t_1, t_2, ...]/(t_i^{p^3} - t_i)$ *and the inner degrees are* $|t_s| \equiv 2(p^s - 1)$ *mod* $2(p^3 - 1)$*. The coproduct structure of* $S(3)$ *is that inherited from* BP_*BP*. In particular,* $\Delta(t_s) = \sum_{k=0}^{s} t_k \otimes t_{s-k}^{p^k}$ *for* $s \leq 3$*, and* $\Delta(t_s) = \sum_{k=0}^{s} t_k \otimes t_{s-k}^{p^k} - \tilde{b}_{s-3,2}$ *for* $s > 3$*.*

Notations 2.6. We let $\tilde{b}_{i,j}$ denote the mod p reduction of $b_{i,j}$ in Notations 2.3.

2.7. **Cobar complexes.** Cobar complexes are helpful in computing certain *Ext* groups, such as $Ext^{*,*}_{BP_*BP}(BP_*, BP_*)$, $Ext^{*,*}_{BP_*BP}(BP_*, v_n^{-1}BP_*/I_n)$, and $Ext^{*,*}_{S(n)}(\mathbb{F}_p, \mathbb{F}_p)$. We now recall the relevant definitions and constructions.

Definition 2.8. Let (A, Γ) be a Hopf algebroid. A *right* Γ*-comodule* M is a right A-module M together with a right A-linear map $\psi : M \to M \otimes_A \Gamma$ which is counitary and coassociative. Left Γ-comodules are defined similarly.

Definition 2.9. Let (A, Γ) be a Hopf algebroid. Let M be a right Γ-comodule. The cobar complex $\Omega^{*,*}_\Gamma(M)$ is a cochain complex with $\Omega^{s,*}_\Gamma(M) = M \otimes_A \overline{\Gamma}^{\otimes s}$, where $\overline{\Gamma}$ is the augmentation ideal of $\varepsilon : \Gamma \to A$. The differentials $d : \Omega^{s,*}_\Gamma(M) \to \Omega^{s+1,*}_\Gamma(M)$ are given by

$$d(m \otimes x_1 \otimes x_2 \otimes \cdots \otimes x_s) = -(\psi(m) - m \otimes 1) \otimes x_1 \otimes x_2 \otimes \cdots \otimes x_s$$
$$- \sum_{i=1}^{s} (-1)^{\lambda_{i,j_i}} m \otimes x_1 \otimes \cdots \otimes x_{i-1} \otimes (\sum_{j_i} x'_{i,j_i} \otimes x''_{i,j_i}) \otimes x_{i+1} \otimes \cdots \otimes x_s$$

where we denote

$$\sum_{j_i} x'_{i,j_i} \otimes x''_{i,j_i} = \Delta(x_i) - 1 \otimes x_i - x_i \otimes 1 \tag{2.7}$$

$$\lambda_{i,j_i} = i + |x_1| + \cdots + |x_{i-1}| + |x'_{i,j_i}|. \tag{2.8}$$

Proposition 2.10 ([17] Section A1.2)**.** *The cohomology of* $\Omega^{s,*}_\Gamma(M)$ *is* $Ext^{s,*}_\Gamma(A, M)$*. Moreover, if* M *is also a commutative associative* A*-algebra such that the structure map* ψ *is an algebra map, then* $Ext^{s,*}_\Gamma(A, M)$ *has a naturally induced product structure.*

3. The cohomology of $S(3)$

The cohomology $H^{*,*}S(3) := Ext^{*,*}_{S(3)}(\mathbb{F}_p, \mathbb{F}_p)$ of the Hopf algebra $S(3)$ has been extensively studied. For $p \geq 5$, Ravenel [16] computed the $\mathbb{F}_p$-module structure of $H^{*,*}S(3)$. The $\mathbb{F}_p$-algebra structure of $H^{*,*}S(3)$ was subsequently computed by Yamaguchi in [21], although there is a typo [3, Remark A.1]. Gu-Wang-Wu [3] recomputed the $\mathbb{F}_p$-algebra structure of $H^{*,*}S(3)$ for $p \geq 7$ using a carefully constructed May spectral sequence.

In this section, we recall the $\mathbb{F}_p$-algebra structure of $H^{*,*}S(3)$ determined in [3], we also take this opportunity to correct some typos in [3].

Theorem 3.1 ([3] Theorem 2.2)**.** *Let* $p \geq 7$ *be a prime number. The Hopf algebra* $S(3)$ *can be given an increasing filtration by setting the May degrees as follows: (i) for* $s = 1, 2, 3$*, let* $M(t_s^{p^j}) = 2s - 1$*, and (ii) for* $s > 3$*,* $j \in \mathbb{Z}/3$*, inductively define* $M(t_s^{p^j}) = max\left\{M(t_k^{p^j}) + M(t_{s-k}^{p^{j+k}}),\ p \cdot M(t_{s-3}^{p^{j+2}}) \mid 0 < k < s\right\} + 1$*. The filtration of* $S(3)$ *naturally induces a filtration of* $\Omega^{*,*}_{S(3)}(\mathbb{F}_p)$*. The associated May spectral sequence (MSS) converges to* $H^{*,*}S(3)$*. The MSS has* E_1*-page*

$$E_1^{*,*,*} = E[h_{i,j} | i \geq 1, j \in \mathbb{Z}/3] \otimes P[b_{i,j} | i \geq 1, j \in \mathbb{Z}/3] \tag{3.1}$$

and $d_r : E_r^{s,t,M} \to E_r^{s+1,t,M-r}$, *where*

$$\begin{aligned} h_{i,j} =& [t_i^{p^j}] \in E_1^{1,2(p^i-1)p^j,*} \\ b_{i,j} =& [\sum_{k=1}^{p-1} \binom{p}{k}/p\ (t_i^{p^j})^k \otimes (t_i^{p^j})^{p-k}\ ] \in E_1^{2,2(p^i-1)p^{j+1},*} \end{aligned} \tag{3.2}$$

Proposition 3.2 ([3] Proposition 3.1)**.** *Let $p \geq 7$ be a prime number. As a $\mathbb{F}_p$-module, $H^{*,*}S(3)$ is isomorphic to $E[\rho] \otimes M$, where $\rho \in H^{1,*}S(3)$, M is a $\mathbb{F}_p$-module with the following generators ($i \in \mathbb{Z}/3$):*

dim0: 1;
dim1: $h_{1,i}$;
dim2: $e_{4,i}, \quad g_i, \quad k_i$;
dim3: $e_{4,i}h_{1,i}, \quad e_{4,i}h_{1,i+1}, \quad g_ih_{1,i+1}, \quad \mu_i, \quad \nu_i, \quad \xi$;
dim4: $e_{4,i}^2, \quad e_{4,i}e_{4,i+1}, \quad e_{4,i}g_{i+1}, \quad e_{4,i}g_{i+2}, \quad e_{4,i}k_i, \quad \theta_i$;
dim5: $e_{4,i}e_{4,i+1}h_{1,i+2}, \quad (e_{4,i}e_{4,i+1}h_{1,i+2} = e_{4,i+1}e_{4,i+2}h_{1,i})$
$e_{4,i}^2h_{1,i+1}, \quad e_{4,i}^2h_{1,i+2}, \quad e_{4,i}\mu_{i+2}, \quad e_{4,i}\nu_i, \quad \eta_i$;
dim6: $e_{4,i}^2e_{4,i+1}, \quad e_{4,i}^2e_{4,i+2}, \quad e_{4,i}e_{4,i+1}g_{i+2}$;
dim7: $e_{4,i}e_{4,i+1}\mu_{i+2}$;
dim8: $e_{4,i}^2e_{4,i+2}g_{i+1}, \quad (e_{4,i}^2e_{4,i+2}g_{i+1} = e_{4,i+1}^2e_{4,i}g_{i+2})$.

Here, the generators can be described via their MSS representatives as follows:

$$\begin{aligned} h_{1,i} &:= t_1^{p^i} & \rho &:= h_{3,0} + h_{3,1} + h_{3,2} \\ e_{4,i} &:= h_{1,i}h_{3,i+1} + h_{2,i}h_{2,i+2} + h_{3,i}h_{1,i} & g_i &:= h_{2,i}h_{1,i} \\ k_i &:= h_{2,i}h_{1,i+1} & \mu_i &= h_{3,i}h_{2,i}h_{1,i} \\ \nu_i &:= h_{3,i}h_{2,i+1}h_{1,i+2} & \xi &= \textstyle\sum_{i=0}^2 h_{3,i}e_{3,i+1} + h_{2,0}h_{2,1}h_{2,2} \\ \theta_i &= h_{3,i}h_{2,i+2}h_{2,i}h_{1,i} & \eta_i &= h_{3,i}h_{3,i+1}h_{2,i+2}h_{2,i}h_{1,i} \end{aligned}$$

Here we denote $h_{i,j} := t_i^{p^j}$ for $j \in \mathbb{Z}/3$, and $e_{3,i} := h_{1,i}h_{2,i+1} + h_{2,i}h_{1,i+2}$ for $i \in \mathbb{Z}/3$.

Remark 3.3. Recall that we have $S(3) \cong \mathbb{F}_p[t_1, t_2, ...]/(t_i^{p^3} - t_i)$, as stated in Proposition 2.5. This implies that $t_i^{p^j} = t_i^{p^{j+3}} \in S(3)$. Consequently, we have $h_{1,i} = h_{1,i+3}$, $e_{4,i} = e_{4,i+3}$, and so on. Therefore, we choose our index to be $i \in \mathbb{Z}/3$ to indicate the equivalences. While it is possible to choose the index as $i = 0, 1, 2$, the current notation system is more natural and concise in expressing the generators and their product relations. Similarly, in Theorem 3.1, we let $j \in \mathbb{Z}/3$ for the same reasons.

Remark 3.4. We have taken the opportunity to correct a typo in [3], where it was incorrectly claimed that $\xi = \sum h_{3,i+1}e_{3,i} + \sum h_{2,i}h_{2,i+1}h_{2,i+2}$. The formula has now been corrected and updated in Proposition 3.2.

The $\mathbb{F}_p$-algebra structure of $H^{*,*}S(3)$ is complicated. However, as we will see in Proposition 4.17, in this paper we only need to care about the product structure of the sub-algebra generated by the elements $\{h_{1,i}, e_{4,i}, k_i, \nu_i | i \in \mathbb{Z}/3\}$. Note for $x \in H^{i,*}S(3)$, $y \in H^{j,*}S(3)$, one can show $x \cdot y = (-1)^{ij}y \cdot x$. We record all nontrivial products of these generators as follows.

Proposition 3.5 ([3] Appendix A)**.** *Let $p \geq 7$ be a prime number. All nontrivial products amongst generators of $H^{*,*}S(3)$ in the set $\{h_{1,i}, e_{4,i}, k_i, \nu_i | i \in \mathbb{Z}/3\}$ can be expressed in terms of the generators in Proposition 3.2 as follows.*

dim3: $e_{4,i} \cdot h_{1,i+2} = e_{4,i+2}h_{1,i}$ $\quad k_i \cdot h_{1,i} = -g_i h_{1,i+1}$

dim4: $e_{4,i} \cdot k_{i+1} = e_{4,i+1}g_{i+2}$ $\quad \nu_i \cdot h_{1,i} = \frac{1}{3}e_{4,i+1}g_{i+2}$

$\nu_i \cdot h_{1,i+1} = \frac{2}{3}e_{4,i+2}g_{i+1} - \frac{1}{3}e_{4,i+1}k_{i+1} - \frac{1}{3}\rho g_{i+1}h_{1,i+2}$

dim5: $e_{4,i}e_{4,i+1} \cdot h_{1,i} = e_{4,i}^2 h_{1,i+1}$ $\quad e_{4,i}e_{4,i+1} \cdot h_{1,i+1} = e_{4,i+1}^2 h_{1,i}$

$\nu_i \cdot k_i = \frac{1}{2}e_{4,i+1}^2 h_{1,i+2}$ $\quad \nu_i k_{i+2} = -\frac{1}{2}e_{4,i+2}^2 h_{1,i}$

$\nu_i \cdot e_{4,i+1} = -e_{4,i+2}\mu_{i+1} + \frac{1}{3}\rho e_{4,i+2}g_{i+1} + \frac{1}{3}\rho e_{4,i+1}k_{i+1}$

dim6: $e_{4,i}h_{1,i} \cdot \nu_i = -\frac{1}{3}e_{4,i}e_{4,i+1}g_{i+2}$ $\quad e_{4,i}h_{1,i+1} \cdot \nu_i = -\frac{1}{3}e_{4,i+2}e_{4,i}g_{i+1}$

$\nu_i \cdot \nu_{i+1} = \frac{1}{3}\rho e_{4,i+2}^2 h_{1,i} - \frac{1}{6}e_{4,i+2}^2 e_{4,i}$ $\quad e_{4,i}k_i \cdot e_{4,i+2} = e_{4,i+1}e_{4,i+2}g_i$

$e_{4,i}^2 \cdot k_{i+1} = e_{4,i}e_{4,i+1}g_{i+2}$

dim7: $e_{4,i}^2 \cdot \nu_i = \frac{2}{3}\rho e_{4,i}e_{4,i+1}g_{i+2} - e_{4,i}e_{4,i+1}\mu_{i+2}$

$e_{4,i}e_{4,i+1} \cdot \nu_i = -e_{4,i+2}e_{4,i}\mu_{i+1} + \frac{2}{3}\rho e_{4,i+2}e_{4,i}g_{i+1}$

dim8: $e_{4,i}^2 \cdot e_{4,i+1}k_{i+1} = e_{4,i}^2 e_{4,i+2}g_{i+1}$ $\quad e_{4,i}^2 h_{1,i+1} \cdot \nu_i = -\frac{1}{3}e_{4,i}^2 e_{4,i+2}g_{i+1}$

dim9: $e_{4,i}\nu_i \cdot e_{4,i}e_{4,i+1} = \frac{1}{3}\rho e_{4,i+2}e_{4,i}^2 g_{i+1}$

Remark 3.6. We take this opportunity to correct a typo in [3]. The formula for $e_{4,i}e_{4,i+1} \cdot \nu_i$ in dimension 7 is now corrected here.

4. Representations of α, β, γ-family elements

In this section, we recall the constructions of the α,β,γ-family elements in the E_2-page $Ext^{*,*}_{BP_*BP}(BP_*, BP_*)$ of the Adams-Novikov spectral sequence. Then we determine their images under the comparison map $\phi : Ext^{*,*}_{BP_*BP}(BP_*, BP_*) \to H^{*,*}(S(3))$.

Note we can write ϕ as the composition of several maps. We have

$$\phi = Ext^{*,*}_{BP_*BP}(BP_*, BP_*) \xrightarrow{\eta} Ext^{*,*}_{BP_*BP}(BP_*, v_3^{-1}BP_*/I_3) \xrightarrow{\psi} H^{*,*}(S(3)) \tag{4.1}$$

with $I_3 = (p, v_1, v_2) \subset BP_*$ and $\psi = \psi_3\psi_2\psi_1$, where

$$\psi_1 : Ext^{*,*}_{BP_*BP}(BP_*, v_3^{-1}BP_*/I_3) \xrightarrow{\cong} Ext^{*,*}_{\Sigma(3)}(K(3)_*, K(3)_*), \tag{4.2}$$

$$\psi_2 : Ext^{*,*}_{\Sigma(3)}(K(3)_*, K(3)_*) \to Ext^{*,*}_{\Sigma(3)}(K(3)_*, K(3)_*) \otimes_{K(3)_*} \mathbb{F}_p, \tag{4.3}$$

$$\psi_3 : Ext^{*,*}_{\Sigma(3)}(K(3)_*, K(3)_*) \otimes_{K(3)_*} \mathbb{F}_p \xrightarrow{\cong} Ext^{*,*}_{S(3)}(\mathbb{F}_p, \mathbb{F}_p) = H^{*,*}(S(3)). \tag{4.4}$$

We will first determine all nontrivial images of the α,β,γ-family elements under the map η. Then, we will determine all nontrivial images of the α,β,γ-family elements under the composition ϕ.

4.1. α-family elements.

Let $n \geq 0$, $p \nmid s \geq 1$. Then $v_1^{sp^n} \in Ext^{0,*}_{BP_*BP}(BP_*, BP_*/p^{n+1})$. We define $\alpha_{sp^n/n+1} := \delta_0(v_1^{sp^n}) \in Ext^{1,*}_{BP_*BP}(BP_*, BP_*)$, where δ_0 is the boundary-homomorphism associated to the short exact sequence

$$0 \to \Omega_{BP_*BP}(BP_*) \xrightarrow{p^{n+1}} \Omega_{BP_*BP}(BP_*) \to \Omega_{BP_*BP}(BP_*/p^{n+1}) \to 0 \tag{4.5}$$

of cobar complexes (Definition 2.9). We often abbreviate $\alpha_{s/1}$ to α_s.

Theorem 4.2 ([14])**.** *Let p be an odd prime. Then $Ext^{1,*}_{BP_*BP}(BP_*, BP_*)$ is generated by $\alpha_{sp^n/n+1}$ for $n \geq 0$, $p \nmid s \geq 1$.*

In order to determine the image of η, we introduce the following notion.

Definition 4.3. Let $n \geq 1$. We define $I[n]$ as the ideal of BP_* generated by monomials $p^i v_1^j v_2^k$ such that $i + j + k = n$. In particular, $I[1] = (p, v_1, v_2) = I_3 \subset BP_*$.

Lemma 4.4. *Let d denote the differential of the cobar complex $\Omega^{*,*}_{BP_*BP}(BP_*)$. Let $x \in I[n] \subset BP_* = \Omega^{0,*}_{BP_*BP}(BP_*)$ for some $n \geq 1$. Then $d(x) \in I[n] \cdot \Omega^{1,*}_{BP_*BP}(BP_*)$.*

Proof. BP_* can be regarded as a right BP_*BP-comodule with $\eta_R : BP_* \to BP_*BP$ as the structure map. According to Definition 2.9, for $x \in BP_*$, we have

$$d(x) = -\psi(x) + x \otimes 1 = -\eta_R(x) + x \otimes 1. \tag{4.6}$$

Note that if $x \in I[n]$, then $x \otimes 1 \in I[n] \cdot \Omega^{1,*}_{BP_*BP}(BP_*)$. Therefore, it is sufficient to show that $\eta_R(x) \in I[n] \cdot \Omega^{1,*}_{BP_*BP}(BP_*)$. Furthermore, by considering each summand separately, we can assume that x is a monomial in $BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \cdots]$. Write $x = p^i v_1^j v_2^k y$, where $i + j + k \geq n$. Using (2.2) and (2.3), we have

$$\begin{aligned}\eta_R(p^i v_1^j v_2^k y) &= \eta_R(p^i)\eta_R(v_1^j)\eta_R(v_2^k)\eta_R(y)\\ &= p^i(v_1 + pt_1)^j(v_2 + v_1 t_1^p + pt_2 + L)^k \eta_R(y)\end{aligned} \tag{4.7}$$

where $L \in (p^2, v_1^p) \cdot \Omega^{1,*}_{BP_*BP}(BP_*)$. By counting the exponents, we can see that $\eta_R(x) \in I[n] \cdot \Omega^{1,*}_{BP_*BP}(BP_*)$.

□

Proposition 4.5. *For the image of the α-family elements, we have*

(1) $\eta(\alpha_1) = -t_1$.
(2) $\eta(\alpha_{sp^n/n+1}) = 0$, for any other $\alpha_{sp^n/n+1}$.

Notations 4.6. In this paper, we often abuse the notation and refer to the elements in $Ext^{s,*}_{\Gamma}(A, M)$ by their representatives in the associated cobar complex $\Omega^{s,*}_{\Gamma}(M)$ when no confusion arises. For example, here we let $-t_1$ denote the element in $Ext^{1,*}_{BP_*BP}(BP_*, v_3^{-1}BP_*/I_3)$ represented by $-t_1 \in \Omega^{1,*}_{BP_*BP}(v_3^{-1}BP_*/I_3)$.

Proof of Proposition 4.5. We will work on the cobar complex level to find explicit expressions for the α-family elements.

Note that $\alpha_1 = \delta_0(v_1)$. By definition of the connecting homomorphism δ_0, we have

$$\delta_0(v_1) = \frac{d(v_1)}{p} = -\frac{(\eta_R(v_1) - v_1 \otimes 1)}{p} = -t_1, \tag{4.8}$$

where we let v_1 also denote the preimage of v_1 with respect to the map $\Omega_{BP_*BP}(BP_*) \to \Omega_{BP_*BP}(BP_*/p^{n+1})$ and let d denote the differential map of the cobar complex $\Omega_{BP_*BP}(BP_*)$. Therefore, upon reduction modulo I_3, we find that $\eta(\alpha_1) = -t_1$. This proves statement (1).

For general α-family elements, we have

$$\alpha_{sp^n/n+1} = \delta_0(v_1^{sp^n}) = \frac{d(v_1^{sp^n})}{p^{n+1}}. \tag{4.9}$$

Note that $v_1^{sp^n} \in I[sp^n]$. By Lemma 4.4, $d(v_1^{sp^n}) \in I[sp^n] \cdot \Omega_{BP_*BP}(BP_*)$. Then

$$\frac{d(v_1^{sp^n})}{p^{n+1}} \in I[sp^n - n - 1] \cdot \Omega_{BP_*BP}(BP_*). \tag{4.10}$$

When $\alpha_{sp^n/n+1} \neq \alpha_1$, we have $n > 0$ or $s > 1$. Then $sp^n - n - 1 \geq 1$. So

$$\alpha_{sp^n/n+1} \in I[sp^n - n - 1] \cdot \Omega_{BP_*BP}(BP_*) \subset I[1] \cdot \Omega_{BP_*BP}(BP_*) = I_3 \cdot \Omega_{BP_*BP}(BP_*). \tag{4.11}$$

Therefore, upon reduction modulo I_3, we find that $\eta(\alpha_{sp^n/n+1}) = 0$. This proves statement (2). □

4.7. **β-family elements.** Let $a_0 = 1$, $a_n = p^n + p^{n-1} - 1$ for $n \geq 1$. Define $x_n \in v_2^{-1}BP_*$ as

$$x_0 = v_2, \tag{4.12}$$

$$x_1 = x_0^p - v_1^p v_2^{-1} v_3, \tag{4.13}$$

$$x_2 = x_1^p - v_1^{p^2-1} v_2^{p^2-p+1} - v_1^{p^2+p-1} v_2^{p^2-2p} v_3, \tag{4.14}$$

$$x_n = x_{n-1}^p - 2v_1^{b_n} v_2^{p^n-p^{n-1}+1}, n \geq 3 \tag{4.15}$$

with $b_n = (p+1)(p^{n-1}-1)$ for $n > 1$. Now, if $s \geq 1$ and $p^i | j \leq a_{n-i}$ with $j \leq p^n$ if $s = 1$, then $x_n^s \in Ext^{0,*}_{BP_*BP}(BP_*, BP_*/(p^{i+1}, v_1^j))$. Define

$$\beta_{sp^n/j,i+1} := \delta'\delta''(x_n^s) \in Ext^{2,*}_{BP_*BP}(BP_*, BP_*) \tag{4.16}$$

where δ' (resp. δ'') is the boundary-homomorphism associated to E' (resp. E'')

$$E' : 0 \to \Omega(BP_*) \xrightarrow{p^{i+1}} \Omega(BP_*) \to \Omega(BP_*/p^{i+1}) \to 0, \tag{4.17}$$

$$E'' : 0 \to \Omega(BP_*/p^{i+1}) \xrightarrow{v_1^j} \Omega(BP_*/p^{i+1}) \to \Omega(BP_*/(p^{i+1}, v_1^j)) \to 0. \tag{4.18}$$

where we let $\Omega(-)$ denote $\Omega_{BP_*BP}(-)$. We often abbreviate $\beta_{sp^n/j,1}$ to $\beta_{sp^n/j}$ and $\beta_{sp^n/1}$ to β_{sp^n}. When we work with β-family elements in practice, we require the indexes (s, n, j, i) to satisfy certain relations as specified in the following theorem.

Theorem 4.8 ([10, 11]). *Let p be an odd prime. $Ext^{2,*}_{BP_*BP}(BP_*, BP_*)$ is the direct sum of cyclic subgroups generated by $\beta_{sp^n/j,i+1}$ for $n \geq 0$, $p \nmid s \geq 1$, $j \geq 1$, $i \geq 0$, subject to: (1) $j \leq p^n$, if $s = 1$, (2) $p^i | j \leq a_{n-i}$, and (3) $a_{n-i-1} < j$, if $p^{i+1} | j$.*

Proposition 4.9. *Let $p \geq 7$ be a prime. For the image of the β-family elements, we have*

1. *$\eta(\beta_1) = -b_{1,0}$.*
2. *$\eta(\beta_2) = 2t_2 \otimes t_1^p + t_1 \otimes t_1^{2p}$.*
3. *$\eta(\beta_{p^n/p^n}) = -b_{1,n}$, for $n \geq 1$.*
4. *$\eta(\beta_{p^n/p^n-1}) = t_1 \otimes t_1^{p^{n+1}}$, for $n \geq 1$.*
5. *$\eta(\beta_{sp^n/j,i+1}) = 0$, for any other $\beta_{sp^n/j,i+1}$.*

Proof. Analogous to the proof of Proposition 4.5, we will work at the cobar complex level to find explicit expressions for the β-family elements.

Note that $\beta_{sp^n/j,i+1} = \delta'\delta''(x_n^s)$, we denote

$$y_{sp^n/j,i+1} := \delta''(x_n^s) = \frac{d'(x_n^s)}{v_1^j} \in \Omega(BP_*/p^{i+1}), \tag{4.19}$$

where we let x_n^s also denote the preimage of x_n^s with respect to the map $\Omega(BP_*/p^{i+1}) \to \Omega(BP_*/(p^{i+1}, v_1^j))$ and let d' denote the differential map of the cobar complex $\Omega(BP_*/p^{i+1})$.

Similarly, using the definition of the connecting homomorphism δ', we have

$$\beta_{sp^n/j,i+1} = \delta'(y_{sp^n/j,i+1}) = \frac{d(y_{sp^n/j,i+1})}{p^{i+1}} \in \Omega(BP_*), \tag{4.20}$$

where we let $y_{sp^n/j,i+1}$ also denote the preimage of $y_{sp^n/j,i+1}$ with respect to the map $\Omega(BP_*) \to \Omega(BP_*/p^{i+1})$ and let d denote the differential map of the cobar complex $\Omega(BP_*)$.

In the following, we will study the behavior of the β-family elements through different cases.

Case 1. When $n = 0$, according to Theorem 4.8, we have $i = 0$ and $j = 1$. In this case, $x_n^s = v_2^s$.

Case 1.1 If $s = 1$, we have

$$y_1 = \delta''(v_2) = \frac{d'(v_2)}{v_1} = -\frac{\eta_R(v_2) - v_2 \otimes 1}{v_1}. \tag{4.21}$$

Using (2.3), we can write $\eta_R(v_2) = v_2 + v_1 t_1^p + pt_2 + L$, where $L \in (p^2, v_1^p) \cdot BP_*BP$. Since $p = 0$ in $\Omega(BP_*/p)$, we can write:

$$y_1 = -\frac{v_1 t_1^p + pt_2 + L}{v_1} = -t_1^p + L_1, \tag{4.22}$$

where $L_1 \in (v_1^{p-1}) \cdot \Omega(BP_*/p) \subset I[p-1] \cdot \Omega(BP_*/p)$. Therefore, we can write:

$$\beta_1 = \delta'(y_1) = \frac{d(y_1)}{p} = -\frac{d(t_1^p)}{p} + \frac{d(L_1)}{p}. \tag{4.23}$$

Using Definition 2.9, we have

$$d(t_1^p) = \Delta(t_1^p) - 1 \otimes t_1^p - t_1^p \otimes 1 = (1 \otimes t_1 + t_1 \otimes 1)^p - 1 \otimes t_1^p - t_1^p \otimes 1 \tag{4.24}$$

This implies that $-d(t_1^p)/p = -b_{1,0}$, as mentioned in Notations 2.3. Let L_1 also denote the preimage of L_1 with respect to the map $\Omega(BP_*) \to \Omega(BP_*/p^{i+1})$. Lemma 4.4 implies that $d(L_1) \in I[p-1] \cdot \Omega(BP_*)$. Therefore, we can conclude that:

$$\frac{d(L_1)}{p} \in I[p-2] \cdot \Omega(BP_*) \subset I[1] \cdot \Omega(BP_*) = I_3 \cdot \Omega(BP_*). \tag{4.25}$$

Therefore, upon reduction modulo I_3, we find that $\eta(\beta_1) = -b_{1,0}$. Thus, this proves statement (1).

Case 1.2 If $s = 2$, analogous to Case 1.1, we have:

$$y_2 = \delta''(v_2^2) = \frac{d'(v_2^2)}{v_1} = -\frac{\eta_R(v_2^2) - v_2^2 \otimes 1}{v_1}. \tag{4.26}$$

In $\Omega(BP_*/p)$, we can write:

$$y_2 = -\frac{(v_2 + v_1 t_1^p + pt_2 + L)^2 - v_2^2 \otimes 1}{v_1} = -2v_2 t_1^p - v_1 t_1^{2p} + L_2, \tag{4.27}$$

where $L_2 \in (v_1^{p-1}) \cdot \Omega(BP_*/p) \subset I[p-1] \cdot \Omega(BP_*/p)$. Then, we have:

$$\beta_2 = \delta'(y_2) = \frac{d(y_2)}{p} = -\frac{2d(v_2 t_1^p)}{p} - \frac{d(v_1 t_1^{2p})}{p} + \frac{d(L_2)}{p}. \tag{4.28}$$

Using Definition 2.9, we find the following:

$$\begin{aligned} -\frac{2d(v_2 t_1^p)}{p} &= -\frac{2}{p}[(-\eta_R(v_2) + v_2 \otimes 1) \otimes t_1^p + v_2(\Delta(t_1^p) - 1 \otimes t_1^p - t_1^p \otimes 1)] \\ &= \frac{2}{p}(v_1 t_1^p + pt_2) \otimes t_1^p - \frac{2v_2}{p}(\Delta(t_1^p) - 1 \otimes t_1^p - t_1^p \otimes 1) \\ &\equiv 2t_2 \otimes t_1^p \mod I_3 \end{aligned} \tag{4.29}$$

$$\begin{aligned} -\frac{d(v_1 t_1^{2p})}{p} &= -\frac{1}{p}[(-\eta_R(v_1) + v_1 \otimes 1) \otimes t_1^{2p} + v_1(\Delta(t_1^{2p}) - 1 \otimes t_1^{2p} - t_1^{2p} \otimes 1)] \\ &\equiv t_1 \otimes t_1^{2p} \mod I_3 \end{aligned} \tag{4.30}$$

Therefore, $\eta(\beta_2) = 2t_2 \otimes t_1^p + t_1 \otimes t_1^{2p}$. This proves statement (2).

Case 1.3 If $s \geq 3$, then we have:

$$y_s = \delta''(v_2^s) = \frac{d'(v_2^s)}{v_1} \in I[s-1] \cdot \Omega(BP_*/p). \tag{4.31}$$

$$\beta_s = \delta'(y_s) = \frac{d(y_s)}{p} \in I[s-2] \cdot \Omega(BP_*) \subset I_3 \cdot \Omega(BP_*). \tag{4.32}$$

Therefore, upon reduction modulo I_3, we find that $\eta(\beta_s) = 0$.

Case 2. Let $n \geq 1$, $i = 0$, and $s = 1$. According to Theorem 4.8, we have $j \leq p^n$. Furthermore, we claim that in $\Omega(BP_*/(p, v_1^j))$, we can express x_n as $v_2^{p^n} + L_n$, where $L_n \in I[2p^n - p^{n-1}] \cdot \Omega(BP_*/(p^{i+1}, v_1^j))$.

If $n = 1$, we have $x_1 = v_2^p - v_1^p v_2^{-1} v_3 = v_2^p \in \Omega(BP_*/(p, v_1^j))$ since $j \leq p$ and $v_1^j = 0 \in \Omega(BP_*/(p, v_1^j))$. If $n = 2$, we have $x_2 = x_1^p - v_1^{p^2-1} v_2^{p^2-p+1} - v_1^{p^2+p-1} v_2^{p^2-2p} v_3 = v_2^{p^2} - v_1^{p^2-1} v_2^{p^2-p+1} \in \Omega(BP_*/(p, v_1^j))$ since $j \leq p^2$ and $p, v_1^j = 0 \in \Omega(BP_*/(p, v_1^j))$. The case for general $n \geq 3$ can be proved analogously.

Consequently, we have

$$y_{p^n/j} = \delta''(x_n) = \frac{d'(x_n)}{v_1^j} = \frac{d'(v_2^{p^n})}{v_1^j} + \frac{d'(L_n)}{v_1^j}. \tag{4.33}$$

$$\beta_{p^n/j} = \delta'(y_{p^n/j}) = \frac{d(y_{p^n/j})}{p} = \frac{1}{p} d\left(\frac{d'(v_2^{p^n})}{v_1^j}\right) + \frac{1}{p} d\left(\frac{d'(L_n)}{v_1^j}\right). \tag{4.34}$$

Note that $L_n \in I[2p^n - p^{n-1}] \cdot \Omega(BP_*/(p^{i+1}, v_1^j))$. Analogous to Lemma 4.4, we have

$$\frac{d'(L_n)}{v_1^j} \in I[2p^n - p^{n-1} - j] \cdot \Omega(BP_*/p), \tag{4.35}$$

$$\frac{1}{p} d\left(\frac{d'(L_n)}{v_1^j}\right) \in I[2p^n - p^{n-1} - j - 1] \cdot \Omega(BP_*) \subset I_3 \cdot \Omega(BP_*). \tag{4.36}$$

Case 2.1. If $j = p^n$, analogous to Case 1.1, we can write:

$$\frac{d'(v_2^{p^n})}{v_1^{p^n}} = -t_1^{p^{n+1}} + L_{p^n}, \tag{4.37}$$

where $L_{p^n} \in I[p^{n+1} - p^n] \cdot \Omega(BP_*/p)$, and that

$$\frac{1}{p} d\left(\frac{d'(v_2^{p^n})}{v_1^{p^n}}\right) \equiv -b_{1,n} \mod I_3. \tag{4.38}$$

Therefore, we find that $\eta(\beta_{p^n/p^n}) = -b_{1,n}$. This proves statement (3).

Case 2.2. If $j = p^n - 1$, analogous to Case 1.2, we have:

$$\frac{d'(v_2^{p^n})}{v_1^{p^n-1}} = -v_1 t_1^{p^{n+1}} + L_{p^n-1}, \tag{4.39}$$

where $L_{p^n-1} \in I[p^{n+1} - p^n + 1] \cdot \Omega(BP_*/p)$, and that

$$\frac{1}{p} d\left(\frac{d'(v_2^{p^n})}{v_1^{p^n-1}}\right) \equiv t_1 \otimes t_1^{p^{n+1}} \mod I_3. \tag{4.40}$$

Therefore, we find that $\eta(\beta_{p^n/p^n-1}) = t_1 \otimes t_1^{p^{n+1}}$. This proves statement (4).

Case 2.3. If $j \le p^n - 2$, then $y_{p^n/j} \in I[2] \cdot \Omega(BP_*/p)$, and $\beta_{p^n/j} \in I[1] \cdot \Omega(BP_*)$. Therefore, upon reduction modulo I_3, we find that $\eta(\beta_{p^n/j}) = 0$.

Case 3. Let $n \ge 1$. Moreover, we require that $i \ge 1$ or $s \ge 2$.

Through direct observation, we can see that $x_n \in I[p^n - p^{n-1}] \cdot \Omega(BP_*/(p^{i+1}, v_1^j))$. From this, we can conclude that $x_n^s \in I[sp^n - sp^{n-1}] \cdot \Omega(BP_*/(p^{i+1}, v_1^j))$, $y_{sp^n/j,i+1} \in I[sp^n - sp^{n-1} - j] \cdot \Omega(BP_*/p^{i+1})$, and $\beta_{sp^n/j,i+1} \in I[sp^n - sp^{n-1} - j - i - 1] \cdot \Omega(BP_*)$.

Based on Theorem 4.8, we can deduce that $p^i \mid j \le p^{n-i} + p^{n-i-1} - 1$. We have:

$$sp^n - sp^{n-1} - j - i - 1 \ge (p-1)p^{n-1} - 2j > (p-1)p^{n-1} - 4p^{n-1} \ge 2, \text{ if } i \ge 1, \tag{4.41}$$

$$sp^n - sp^{n-1} - j - i - 1 \ge s(p-1)p^{n-1} - (p+1)p^{n-1} \ge p - 3 \ge 4, \text{ if } i = 0, s \ge 2. \tag{4.42}$$

In any case, we always have $sp^n - sp^{n-1} - j - i - 1 \ge 1$. Therefore, we find that $\eta(\beta_{sp^n/j,i+1}) = 0$. Thus, by combining this case with Case 1.3 and 2.3, we have proven statement (5). □

4.10. **γ-family elements.** Let $s_1 = r_1 p^{e_1}$, $s_2 = r_2 p^{e_2}$, $s_3 = r_3 p^{e_3}$ with p^{e_i} being the largest power of p dividing s_i. For $1 \le s_1 \le p^{e_2}$, $1 \le s_2 \le p^{e_3}$, $1 \le s_3$, one can show $v_3^{s_3}$ is a cycle in $\Omega_{BP_*BP}(BP_*/(p, v_1^{s_1}, v_2^{s_2}))$. Define $\gamma_{s_3/s_2,s_1} := \delta_0\delta_1\delta_2(v_3^{s_3}) \in Ext^{3,*}_{BP_*BP}(BP_*, BP_*)$, where δ_0 (resp. δ_1, δ_2) is the boundary-homomorphism associated to E_0 (resp. E_1, E_2)

$$E_0 : 0 \to \Omega(BP_*) \xrightarrow{p} \Omega(BP_*) \to \Omega(BP_*/p) \to 0, \tag{4.43}$$

$$E_1 : 0 \to \Omega(BP_*/p) \xrightarrow{v_1^{s_1}} \Omega(BP_*/p) \to \Omega(BP_*/(p, v_1^{s_1})) \to 0, \tag{4.44}$$

$$E_2 : 0 \to \Omega(BP_*/(p, v_1^{s_1})) \xrightarrow{v_2^{s_2}} \Omega(BP_*/(p, v_1^{s_1})) \to \Omega(BP_*/(p, v_1^{s_1}, v_2^{s_2})) \to 0. \tag{4.45}$$

where we let $\Omega(-)$ denote $\Omega_{BP_*BP}(-)$. We often abbreviate $\gamma_{s_3/s_2,1}$ to γ_{s_3/s_2} and $\gamma_{s_3/1}$ to γ_{s_3}.

Theorem 4.11 ([10] Corollary 7.8)**.** *We have $0 \ne \gamma_{s_3/s_2,s_1} \in Ext^{3,*}_{BP_*BP}(BP_*, BP_*)$ unless $s_1 < s_2 = p^{e_3} = s_3$. In fact, these elements are linearly independent.*

For the purpose of this paper, we only need to consider γ_s for $s \ge 1$. Recall the following result concerning $\eta(\gamma_s)$.

Proposition 4.12 ([3] Lemma 4.1)**.** *Let $s \ge 1$, we have*

$$\begin{aligned}\eta(\gamma_s) =& s(s-1)v_3^{s-2}(b_{2,0}t_1^{p^2} - t_2^p b_{1,1}) + \frac{s(s-1)}{2}v_3^{s-2}(b_{1,0} \otimes t_1^{2p^2} - 2t_1^p \otimes b_{1,1}(1 \otimes t_1^{p^2} + t_1^{p^2} \otimes 1)) \\ &- s(s-1)(s-2)v_3^{s-3}t_3 \otimes t_2^p \otimes t_1^{p^2}.\end{aligned}$$

Remark 4.13. Here, the result for $\eta(\gamma_s)$ differs from the formula in [3] by a negative sign, as our definitions of the differential in the cobar complex (Definition 2.9) differ by a negative sign.

4.14. **Nontrivial images of ϕ.**

Notations 4.15. Let $G = \{\alpha_{sp^n/n+1}, \beta_{sp^n/j,i+1}, \gamma_s\} \subset Ext^{*,*}_{BP_*BP}(BP_*, BP_*)$ denote the set of all α, β, γ-family elements of the indicated forms.

In Propositions 4.5, 4.9, 4.12, we have determined the images of the elements in G under the map η. Recall from (4.1) that $\phi = \psi \circ \eta$. By direct computation and comparison with $H^*S(3)$ (see Proposition 3.2), we can determine the images of these elements under the map ϕ.

Lemma 4.16. *Let $n \ge 0$, then $\psi(b_{1,n}) = e_{4,n+1} \in H^{*,*}S(3)$.*

Proof. On the level of cobar complexes, the effect of ψ is reduction mod p, sending all v_i with $i \neq 3$ to 0, and sending v_3 to 1. Hence we have $\psi(b_{1,n}) = \tilde{b}_{1,n}$ following Notations 2.6.

By Proposition 2.5, in the cobar complex $\Omega^{*,*}_{S(3)}(\mathbb{F}_p)$, we have $d(t_4) = t_1 \otimes t_3^p + t_2 \otimes t_2^{p^2} + t_3 \otimes t_1^{p^3} - \tilde{b}_{1,2}$. Hence we have equivalent cohomology classes $[\tilde{b}_{1,2}] = [t_1 \otimes t_3^p + t_2 \otimes t_2^{p^2} + t_3 \otimes t_1^{p^3}] = e_{4,3}$. This implies $\psi(b_{1,2}) = e_{4,3}$.

Note that if a is not a multiple of p, then $a^p \equiv a$ modulo p. Hence, working over $\mathbb{F}_p$, we have $\tilde{b}_{1,n+1} = \tilde{b}^p_{1,n}$. Moreover, note that $t_1^{p^3} = t_1$ in $S(3)$, so we have $b_{1,n+3} = b_{1,n}$. Similarly, one can show that $e_{4,n+1} = e^p_{4,n}$ and $e_{4,n+3} = e_{4,n}$. Hence, we conclude that $\psi(b_{1,n}) = e_{4,n+1}$ for each $n \geq 0$.

□

Proposition 4.17. *Under the comparison map $\phi : Ext^{*,*}_{BP_*BP}(BP_*, BP_*) \to H^{*,*}(S(3))$, the nonzero images of elements in G are listed as follows:*

(1) $\phi(\alpha_1) = -h_{1,0}$,
(2) $\phi(\beta_1) = -e_{4,1}$,
(3) $\phi(\beta_2) = 2k_0$,
(4) $\phi(\beta_{p^n/p^n}) = -e_{4,n+1}$, for $n \geq 1$,
(5) $\phi(\gamma_s) = -s(s^2-1)v_0 + s(s-1)\rho k_1$, for $s \not\equiv 0, 1$ mod p.

Proof. We only need to consider the elements in G which have nontrivial images under η. On the level of cobar complexes, the effect of ψ is reduction mod p, sending all v_i with $i \neq 3$ to 0, and sending v_3 to 1.

According to Proposition 4.5, we have $\eta(\alpha_1) = -t_1$. By Proposition 3.2, $-t_1$ represents $-h_{1,0}$ in $H^{*,*}(S(3))$. Therefore, $\phi(\alpha_1) = -h_{1,0}$, which proves statement (1).

Based on Proposition 4.9, we find that $\eta(\beta_1) = -b_{1,0}$ and $\eta(\beta_{p^n/p^n}) = -b_{1,n}$ for $n \geq 1$. Then, Lemma 4.16 implies $\phi(\beta_1) = -e_{4,1}$ and $\phi(\beta_{p^n/p^n}) = -e_{4,n+1}$ for $n \geq 1$. Consequently, statement (2) and (4) are proven.

By Proposition 4.9, we have $\eta(\beta_2) = 2t_2 \otimes t_1^p + t_1 \otimes t_1^{2p}$, and $\eta(\beta_{p^n/p^n-1}) = t_1 \otimes t_1^{p^{n+1}}$, for $n \geq 1$. By computing May degrees (Theorem 3.1), we observe that $2t_2 \otimes t_1^p + t_1 \otimes t_1^{2p}$ has a May filtration leading term of $2t_2 \otimes t_1^p$. According to Proposition 3.2, $2t_2 \otimes t_1^p$ represents $2h_{2,0}h_{1,1} = 2k_0$ in $H^{*,*}(S(3))$. Thus, $\phi(\beta_2) = 2k_0$, confirming statement (3). On the other hand, $t_1 \otimes t_1^{p^{n+1}}$ represents $h_{1,0}h_{1,n+1}$ in $H^{*,*}(S(3))$. By Proposition 3.5, there are no nontrivial products of this form in $H^{*,*}(S(3))$. In other words, $h_{1,0}h_{1,n+1} = 0$ in $H^{*,*}(S(3))$. Hence, $\phi(\beta_{p^n/p^n-1}) = 0$.

The computation of statement (5) is done in Lemma 4.1 of [3]. It's important to note that our result listed here differs from the formula in [3] due to the negative sign discrepancy in the definitions of the cobar complex differential (Definition 2.9).

□

5. Detection of nontrivial products via the cohomology of $S(3)$

In this section, we will utilize the $\mathbb{F}_p$-algebra structure of $H^{*,*}S(3)$ to identify nontrivial products in $Ext^{*,*}_{BP_*BP}(BP_*, BP_*)$. We will then proceed to prove Theorems 1.1 and 1.2.

Proposition 5.1. *Let $p \geq 7$ be a prime. We consider the products of elements in G. Among all such products, only the following ones have a nontrivial image under the comparison map $\phi : Ext^{*,*}_{BP_*BP}(BP_*, BP_*) \to H^{*,*}(S(3))$.*

dim3: $\alpha_1\beta_1$

	$\alpha_1\beta_2$	
	$\alpha_1\beta_{p^n/p^n}$	
dim4:	$\alpha_1\gamma_s$,	$s \not\equiv 0, \pm 1$ *mod p*
	$\beta_1\beta_{p^n/p^n}$	
	$\beta_2\beta_{p^n/p^n}$	$n \not\equiv 0$ *mod 3*
	$\beta_{p^n/p^n}\beta_{p^m/p^m}$	
dim5:	$\beta_1\gamma_s$,	$s \not\equiv 0, 1$ *mod p*
	$\beta_2\gamma_s$,	$s \not\equiv 0, \pm 1$ *mod p*
	$\beta_{p^n/p^n}\gamma_s$,	$s \not\equiv 0, \pm 1$ *mod p and* $n \not\equiv 1$ *mod 3*
	$\alpha_1\beta_1\beta_{p^n/p^n}$,	$n \not\equiv 0$ *mod 3*
	$\alpha_1\beta_{p^n/p^n}\beta_{p^m/p^m}$,	$m \equiv n \not\equiv 2$ *or* $m \not\equiv n$ *mod 3*
dim6:	$\alpha_1\beta_1\gamma_s$,	$s \not\equiv 0, \pm 1$ *mod p*
	$\alpha_1\beta_{p^n/p^n}\gamma_s$,	$s \not\equiv 0, \pm 1$ *mod p and* $n \not\equiv 1$ *mod 3*
	$\beta_1^2\beta_{p^n/p^n}$	$n \not\equiv 0$ *mod 3*
	$\beta_1\beta_{p^n/p^n}\beta_{p^m/p^m}$	$n \equiv m \not\equiv 0$ *or* $n \equiv 0 \not\equiv m$ *mod 3*
	$\beta_{p^n/p^n}\beta_{p^m/p^m}\beta_{p^k/p^k}$	$n \equiv m \not\equiv k$ *mod 3*
	$\beta_2\beta_{p^n/p^n}\beta_{p^m/p^m}$,	$n \equiv 1, m \not\equiv 0$ *mod 3*
dim7:	$\beta_1\beta_{p^n/p^n}\gamma_s$,	$s \not\equiv 0, \pm 1$ *mod p and* $n \equiv 2$ *mod 3*
	$\beta_{p^n/p^n}\beta_{p^m/p^m}\gamma_s$,	$s \not\equiv 0, \pm 1$ *mod p and* $n \equiv 2, m \not\equiv 1$ *mod 3*
dim8:	$\alpha_1\beta_1\beta_{p^n/p^n}\gamma_s$,	$s \not\equiv 0, \pm 1$ *mod p and* $n \equiv 2$ *mod 3*
	$\alpha_1\beta_{p^n/p^n}\beta_{p^m/p^m}\gamma_s$,	$s \not\equiv 0, \pm 1$ *mod p and* $n \equiv 0, m \equiv 2$ *mod 3*
	$\beta_2\beta_{p^n/p^n}\beta_{p^m/p^m}\beta_{p^k/p^k}$,	$n \equiv 2, m \equiv k \equiv 1$ *mod 3*

Proof. This follows from a straightforward computation using Propositions 3.2, 3.5 and 4.17. We provide a detailed treatment of the dimension 3 case to illustrate the idea.

Let $x = x_1 x_2 \cdots x_m$ be a product of elements in G such that $\phi(x) \neq 0$. Then we have $\phi(x_i) \neq 0$ for $1 \leq i \leq m$. By Proposition 4.17, this implies $x_i \in \{\alpha_1, \beta_1, \beta_2, \beta_{p^n/p^n}, \gamma_s | n \geq 1, s \not\equiv 0, 1 \text{ mod } p\}$.

If x has dimension 3, there are several possibilities: $x = \alpha_1\beta_1$, $x = \alpha_1\beta_2$, $x = \alpha_1\beta_{p^n/p^n}$, or $x = \alpha_1^3$. By Propositions 3.2, 3.5, and 4.17, we have:

(1) $\phi(\alpha_1\beta_1) = h_{1,0}e_{4,1} = h_{1,1}e_{4,0} \neq 0$.
(2) $\phi(\alpha_1\beta_2) = -2h_{1,0}k_0 = 2h_{1,1}g_0 \neq 0$.
(3) $\phi(\alpha_1\beta_{p^n/p^n}) = h_{1,0}e_{4,n+1} \neq 0$.
(4) $\phi(\alpha_1^3) = -h_{1,0}^3 = 0$.

Therefore, only the first three cases can be detected as nontrivial products. This proves the statement in dimension 3.

As a further example to illustrate the computations, we show that for $s \not\equiv 0, \pm 1$ mod p and $n \equiv 2$ mod 3, $\alpha_1\beta_1\beta_{p^n/p^n}\gamma_s$ is a nontrivial product in $Ext^{8,*}_{BP_*BP}(BP_*, BP_*)$. This result will be used in the proof of Theorem 1.1.

By Propositions 3.2, 3.5, and 4.17, we have:

$$\begin{aligned}
\phi(\alpha_1\beta_1\beta_{p^n/p^n}\gamma_s) &= (-h_{1,0})(-e_{4,1})(-e_{4,n+1})(-s(s^2-1)v_0 + s(s-1)\rho k_1) \\
&= s(s^2-1)h_{1,0}e_{4,1}e_{4,n+1}v_0 - s(s-1)h_{1,0}e_{4,1}e_{4,n+1}\rho k_1 \\
&= s(s^2-1)(h_{1,0}v_0)e_{4,1}e_{4,0} \\
&= -\frac{s(s^2-1)}{3}e_{4,1}g_2e_{4,1}e_{4,0} \\
&\neq 0
\end{aligned}$$

Therefore, $\alpha_1\beta_1\beta_{p^n/p^n}\gamma_s$ is a nontrivial product detected by the map ϕ.

□

Now we proceed to study nontrivial products in the stable homotopy ring of the sphere $\pi_*(S)$. Let $p \geq 7$, $s \geq 1$. It is proved in [10, 14, 18, 19] that $\alpha_s, \beta_s, \gamma_s$ all represent nontrivial elements in $\pi_*(S)$. Using the Adams spectral sequence, Cohen [2] also found another family of nontrivial elements $\zeta_n \in \pi_*(S)$, for $n \geq 1$. Cohen [2] shows that, in the Adams-Novikov spectral sequence, ζ_n is represented by $\alpha_1\beta_{p^n/p^n} + \alpha_1 x \in Ext^{3,*}_{BP_*BP}(BP_*, BP_*)$, where $x = \sum_{s,k,j} a_{s,k,j}\beta_{sp^k/j}$, $0 \leq a_{s,k,j} \leq p-1$, and $a_{1,n,p^n} = 0$. Moreover, [3] shows $s \geq 2$ by comparing inner degrees.

Proof of Theorem 1.1. The representation of $\zeta_n\beta_1\gamma_s$ on the E_2-page of the ANSS is $(\alpha_1\beta_{p^n/p^n} + \alpha_1 x)\beta_1\gamma_s \in Ext^{8,*}_{BP_*BP}(BP_*, BP_*)$. According to Proposition 4.17, we have $\phi(x) = 0$. Furthermore, based on Proposition 5.1, we have $\phi((\alpha_1\beta_{p^n/p^n} + \alpha_1 x)\beta_1\gamma_s) = \phi(\alpha_1\beta_{p^n/p^n}\beta_1\gamma_s) \neq 0$. Hence, $(\alpha_1\beta_{p^n/p^n} + \alpha_1 x)\beta_1\gamma_s \neq 0 \in Ext^{8,*}_{BP_*BP}(BP_*, BP_*)$. It is worth noting that $\alpha_1\beta_{p^n/p^n} + \alpha_1 x$, β_1, and γ_s are all permanent cycles in the ANSS. Consequently, their product is also a permanent cycle.

We observe that the differentials of the ANSS have the form $d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}$, where $r \geq 2$. Additionally, the inner degrees of the elements in the ANSS are all multiples of $q = 2p-2$. Thus, the first potentially nontrivial differentials in the ANSS occur at d_{2p-1}. Considering the degrees, $(\alpha_1\beta_{p^n/p^n} + \alpha_1 x)\beta_1\gamma_s \in Ext^{8,*}_{BP_*BP}(BP_*, BP_*)$ cannot be the image of any differential. Consequently, $(\alpha_1\beta_{p^n/p^n} + \alpha_1 x)\beta_1\gamma_s$ represents nontrivial products $\zeta_n\beta_1\gamma_s \in \pi_*(S)$.

□

Proof of Theorem 1.2. Let x be a product in $\pi_*(S)$ where each factor belongs to the set $\{\alpha_s, \beta_s, \gamma_s, \zeta_s | s \geq 1\}$. Let $y \in Ext^{*,*}_{BP_*BP}(BP_*, BP_*)$ represent x on the Adams-Novikov E_2-page. If x can be detected as nontrivial by comparing with $H^*S(3)$, then we have $\phi(y) \neq 0 \in H^{*,*}(S(3))$. Since all possible forms of y are listed in Proposition 5.1, we conclude that x must have one of the nine forms listed in the theorem.

On the other hand, assuming x has one of the given forms listed in the theorem, we can show that x is nontrivial using a similar proof to the one in Theorem 1.1.

□

References

[1] Marc Aubry. Calculs de groupes d'homotopie stables de la sphère, par la suite spectrale d'Adams-Novikov. *Math. Z.*, 185(1):45–91, 1984.

[2] Ralph L. Cohen. *Odd primary infinite families in stable homotopy theory*, volume 30. American Mathematical Soc., 1981.

[3] Xing Gu, Xiangjun Wang, and Jianqiu Wu. The composition of R. Cohen's elements and the third periodic elements in stable homotopy groups of spheres. *Osaka J. Math.*, 58(2):367–382, 2021.

[4] Michiel Hazewinkel. A universal formal group and complex cobordism. *Bull. Amer. Math. Soc.*, 81(5):930–933, 1975.

[5] Ryo Kato and Katsumi Shimomura. Products of Greek letter elements dug up from the third Morava stabilizer algebra. *Algebraic & Geometric Topology*, 12(2):951–961, 2012.

[6] Chun-Nip Lee and Douglas C Ravenel. On the nilpotence order of β_1. *Mathematical Proceedings of the Cambridge Philosophical Society*, 115(3):483–488, 1994.

[7] C.N. Lee. Detection of some elements in the stable homotopy groups of spheres. *Math.Z*, 222:231–246, 1996.

[8] Xiugui Liu and Jiayi Liu. On the product $\alpha_1\beta_1^2\beta_2\gamma_s$ in the stable homotopy groups of spheres. *Topology and its Applications*, 322:108331, 2022.

[9] Haynes R. Miller and Douglas C. Ravenel. Morava stabilizer algebras and the localization of Novikov's E_2-term. *Duke Math. J.*, 44(2):433–447, 1977.
[10] Haynes R. Miller, Douglas C. Ravenel, and W. Stephen Wilson. Periodic phenomena in the Adams-Novikov spectral sequence. *Ann. of Math. (2)*, 106(3):469–516, 1977.
[11] Haynes R Miller and W. Stephen Wilson. On Novikov's Ext^1 modulo an invariant prime ideal. *Topology*, 15(2):131–141, 1976.
[12] Hirofumi Nakai. The chromatic E_1-term $H_0M_1^2$ for $p > 3$. *New York J. Math*, 6(21):54, 2000.
[13] Hirofumi Nakai. The chromatic E_1-term $H_0M_1^2$ for $p = 3$. *Mem. Fac. Sci. Kochi Univ. (Math)*, 23(27):44, 2002.
[14] S. P. Novikov. Methods of algebraic topology from the point of view of cobordism theory. *Izv. Akad. Nauk SSSR Ser. Mat.*, 31:855–951, 1967.
[15] Douglas C. Ravenel. The structure of Morava stabilizer algebras. *Invent. Math.*, 37(2):109–120, 1976.
[16] Douglas C. Ravenel. The cohomology of the Morava stabilizer algebras. *Math. Z.*, 152(3):287–297, 1977.
[17] Douglas C. Ravenel. *Complex cobordism and stable homotopy groups of spheres*, volume 121 of *Pure and Applied Mathematics*. Academic Press, Inc., Orlando, FL, 1986.
[18] Larry Smith. On realizing complex bordism modules. *Amer. J. Math.*, 92:793–856, (1970).
[19] H. Toda. p-primary components of homotopy groups. IV. Compositions and toric constructions. *Mem. Coll. Sci. Univ. Kyoto Ser. A. Math.*, 32:297–332, 1959.
[20] Xiangjun Wang, Yaxing Wang, and Yu Zhang. Some nontrivial secondary Adams differentials on the fourth line. *New York J. Math*, 29:687–707, 2023.
[21] Atsushi Yamaguchi. The structure of the cohomology of Morava stabilizer algebra $S(3)$. *Osaka J. Math.*, 29(2):347–359, 1992.
[22] Hao Zhao, Xiangjun Wang, and Linan Zhong. The convergence of the product $\overline{\gamma}_{p-1}h_0b_{n-1}$ in the Adams spectral sequence. *Forum Math.*, 27(3):1613–1637, 2015.