An Introduction to Motivic Homotopy Theory part 5: stable motivic homotopy theory

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Introduction

The purpose of this talk is to introduce the basic setting and constructions for stable motivic homotopy theory. We will also talk about some basic computational phenomina regarding motivic Steenrod algebra and motivic Adams spectral sequence.

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In particular, $\mathcal{O}_{Spec(R)}(Spec(R)) = \mathcal{O}_{Spec(R)}(D(1)) = R_1 = R$, where 1 is the unit in R. Besides, by definition of sheaves, the value of a sheaf on empty set is always the terminal object. In our situation, we get $\mathcal{O}_{Spec(R)}(\emptyset) = 0$, the zero ring.

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Example 2: $\mathbb{A}^1_{\mathbb{C}} :=$ Spec $\mathbb{C}[x]$. The prime ideals of $\mathbb{C}[x]$ are (0) and $(x - a)$, where $a \in \mathbb{C}$. We can mostly picture $\mathbb{A}^1_{\mathbb{C}}$ as $\mathbb{C}$: the point $(x - a)$ we will correspond to $a \in \mathbb{C}$. We picture the point (0) somewhere on the complex line, but nowhere in particular.

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Example 3: The scheme $\mathbb{A}^1_{\mathbb{C}} - 0$ is $\mathbb{A}^1_{\mathbb{C}}$ restricted to the open subspace *Spec* $\mathbb{C}[x] - (x - 0) = D(x)$. We have $\mathcal{O}_{\mathbb{A}^1_{\mathbb{C}}}(D(x)) = \mathbb{C}[x]_x = \mathbb{C}[x, 1/x]$. Indeed, one can show $\mathbb{A}^1_{\mathbb{C}} - 0 = Spec(\mathbb{C}[x, 1/x])$.

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Example 4: Now we glue together two copies of the affine line $\mathbb{A}^1_{\mathbb{C}}$. Let $X = Spec\ \mathbb{C}[t]$, and $Y = Spec\ \mathbb{C}[u]$. Let $U = D(t) = Spec\ \mathbb{C}[t, 1/t] \subset X$ and $V = D(u) = Spec\ \mathbb{C}[u, 1/u] \subset Y$. We will glue X and Y together along the subspaces U and V. Consider the isomorphism $U \cong V$ via the isomorphism $\mathbb{C}[t, 1/t] \cong \mathbb{C}[u, 1/u]$ given by $t \leftrightarrow 1/u$. The resulting scheme is called the projective line over the field $\mathbb{C}$, and is denoted $\mathbb{P}^1_{\mathbb{C}}$. The scheme $\mathbb{P}^1_{\mathbb{C}}$ is not affine.

Smash product of pointed $\mathbb{C}$-spaces

Note $Id : Spec(\mathbb{C}) \to Spec(\mathbb{C})$ is a terminal object in the category of $\mathbb{C}$-schemes (schemes over $Spec(\mathbb{C})$). We regard $Id : Spec(\mathbb{C}) \to Spec(\mathbb{C})$ as a $\mathbb{C}$-space, still call it $Spec(\mathbb{C})$, via Yoneda embedding.

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A *pointed* $\mathbb{C}$*-space* consists of a $\mathbb{C}$-space X together with a map $Spec(k) \to X$. Let $Spc_*(\mathbb{C})$ denote the category of pointed $\mathbb{C}$-spaces. If X is a $\mathbb{C}$-space, let X_+ denote the canonically pointed space $X \coprod Spec(k)$.

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Let X, Y be pointed $\mathbb{C}$-spaces. Then their smash product $X \wedge Y$ is defined as the pointed $\mathbb{C}$-space associated to the functor

$$M \to X(M) \wedge Y(M)$$

We can see $Spec(\mathbb{C})_+$ is a unit for the smash product.

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Recall we have a pushout square

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Since $\mathbb{A}^1_{\mathbb{C}}$ is motivicly contractible, we get $\mathbb{P}^1_{\mathbb{C}} \sim \Sigma(\mathbb{A}^1_{\mathbb{C}} - 0) = S^{1,0} \wedge S^{1,1} = S^{2,1}$. Here, note in the functor category colimits are computed levelwise. Hence suspension is in the simplicial direction.

Stable motivic homotopy category

A $\mathbb{C}$-*spectrum* is a sequence of pointed $\mathbb{C}$-spaces $X = (X_n)_{n \in N}$ equipped with pointed structure maps $\sigma : \mathbb{P}^1_{\mathbb{C}} \wedge X_n \to X_{n+1}$.

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A *map of* $\mathbb{C}$*-spectra* $f : X \to Y$ is a sequence of maps $X_n \to Y_n$ commuting with structure maps in the obvious way:

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The resulting category of $\mathbb{C}$-spectra is denoted $Spt(\mathbb{C})$.
The sphere spectrum is $\mathbb{S} = (S^{0,0}, S^{2,1}, \cdots)$.

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For $m, n \in \mathbb{Z}$, $X \in Spt(\mathbb{C})$ and $M \in Sm/\mathbb{C}$, define $\pi_{m,n}X(M)$ to be the colimit of the sequence

$$[S^{m,n} \wedge M_+, X_0] \to [S^{m+2,n+1} \wedge M_+, X_1] \to [S^{m+4,n+2} \wedge M_+, X_2] \to \cdots$$

where M_+ denotes M (regarded as a $\mathbb{C}$-space) with a disjoint base-point and $[-,-] = Hom_{H(\mathbb{C})}(-,-)$ denotes motivic homotopy classes of maps. This makes $\pi_{m,n}X$ into a presheaf on $Sm/\mathbb{C}$ called the *presheaf of stable motivic homotopy groups* of X.

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A map $X \to Y$ in $Spt(\mathbb{C})$ is a *stable equivalence* iff it induces isomorphisms of presheaves $\pi_{m,n}X \to \pi_{m,n}Y$ for all $m, n \in \mathbb{Z}$.

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Jardine shows that $Spt(\mathbb{C})$ has a simplicial model category structure where the weak equivalences are defined to be stable equivalences.
The homotopy category of $Spt(\mathbb{C})$ is denoted by $SH(\mathbb{C})$.

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2. (Morel) $\pi_{n,n}\mathbb{S} = K^{MW}_{-n}\mathbb{C}$, where $K^{MW}_{*}\mathbb{C}$ is the Milnor-Witt K-theory of $\mathbb{C}$.
3. (Levine) $\pi^{\mathbb{C}}_{m,0}\mathbb{S} = \pi_m\mathbb{S}$. Moreover, there is a fully faithful embedding functor $c : SH \to SH(\mathbb{C})$ from classical stable homotopy category to motivic stable homotopy category, derived from the constant presheaf functor from pointed spaces to presheaves of pointed spaces over $Sm/\mathbb{C}$.

Geometric realization

Let $X \in Sm/\mathbb{C}$ be a $\mathbb{C}$-scheme. We have a structure map $X \to Spec(\mathbb{C})$. A $\mathbb{C}$-valued point of X is a map $Spec(\mathbb{C}) \to X$ making the composite $Spec(\mathbb{C}) \to X \to Spec(\mathbb{C})$ equal to identity. One writes $X(\mathbb{C})$ for the set of $\mathbb{C}$-points of X.

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For example, consider $X = \mathbb{A}^n_{\mathbb{C}} = Spec(\mathbb{C}[x_1, ..., x_n])$. A $\mathbb{C}$-valued point of X corresponds to a map $\mathbb{C}[x_1, ..., x_n] \to \mathbb{C}$ making $\mathbb{C} \to \mathbb{C}[x_1, ..., x_n] \to \mathbb{C} = id$ which again corresponds to n-tuples of complex numbers.

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The set $X(\mathbb{C})$ can be turned into a topological space by giving it the analytic topology, which is locally induced from the topology of the complex numbers $\mathbb{C}^n$. In particular, we have $\mathbb{A}^n_{\mathbb{C}}(\mathbb{C}) \sim \mathbb{C}^n$, $(\mathbb{A}^1_{\mathbb{C}} - 0)(\mathbb{C}) \sim \mathbb{C}^1 - 0 \sim S^1$.

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This functor $R_{\mathbb{C}}$ is uniquely determined (up to homotopy) by the following properties:

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In the following, if X is a motivic spectrum, we will also let $X(\mathbb{C})$ denote its topological realization $R_{\mathbb{C}}(X)$.

Motivic cohomology

Given a $\mathbb{C}$-spectrum E and smooth $\mathbb{C}$-scheme U, define the (m, n)-th E-cohomology of U to be

$$E^{m,n}(U) = [U_+, \Sigma^{m,n} E]$$

where $\Sigma^{m,n}$ is the (m, n)-th suspension functor $S^{m,n} \wedge -$ on $Spt(\mathbb{C})$.

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When evaluated on the sphere spectrum $\mathbb{S}$, we often write $E^{m,n} := E^{m,n}(\mathbb{S})$, $E_{m,n} := E_{m,n}(\mathbb{S})$.

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Motivic cohomology with integer coefficients is represented in $SH(\mathbb{C})$ by a $\mathbb{C}$-spectrum $H\mathbb{Z}^{\text{mot}}$. Motivic cohomology with mod p coefficients is represented in $SH(\mathbb{C})$ by a $\mathbb{C}$-spectrum $H\mathbb{F}_p^{\text{mot}}$.

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$$H_{*,*} = H_{*,*}(\mathbb{S}) = \mathbb{F}_p[\tilde{\tau}]$$

where $\tilde{\tau}$ is the dual of τ with bidegree (0, -1). However, we often also write $\tilde{\tau}$ as τ.

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One can construct bistable cohomological operations:

$$P^i : H^{*,*}(X) \to H^{*+2i(p-1),*+i(p-1)}(X)$$

where $P^0 = Id$ and $P^n = 0$ if $n < 0$.

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We denote by β the Bockstein homomorphism

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Motivic cohomology

One can construct bistable cohomological operations:

$$P^i : H^{*,*}(X) \to H^{*+2i(p-1),*+i(p-1)}(X)$$

where $P^0 = Id$ and $P^n = 0$ if $n < 0$.
We denote by β the Bockstein homomorphism

$$\beta : H^{*,*}(X) \to H^{*+1,*}(X)$$

$H^{*,*} = \mathbb{F}_p[\tau]$ is concentrated entirely in topological degree 0. It follows that the cohomological operations (other than the identity) act trivially on $H^{*,*}$ for dimension reasons.

Motivic cohomology

We also have the analogous construction of motivic Steenrod algebra $\mathcal{A}^{*,*} = H^{*,*}(H)$ as well as its dual $\mathcal{A}_{*,*} = H_{*,*}(H)$.

Motivic cohomology

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The (graded commutative with respect to the first grading) algebra $\mathcal{A}_{*,*}$ over $H^{*,*}$ is canonically isomorphic to the graded commutative algebra with generators $\xi_i \in \mathcal{A}_{2p^i-2,p^i-1}$, $\tau_i \in \mathcal{A}_{2p^i-1,p^i-1}$ and relations $\xi_0 = 1, \tau_i^2 = 0$. In other words,

$$\mathcal{A}_{*,*} = \mathbb{F}_p[\tau] \otimes \mathbb{F}_p(\xi_1, \xi_2, \cdots) \otimes E(\tau_0, \tau_1, \cdots)$$

where τ has degree (0, -1).

Motivic Adams spectral sequence

Starting with the motivic sphere spectrum $\mathbb{S}$, let H denote $H\mathbb{F}_p^{\mathrm{mot}}$, let $\bar{H}$ be the homotopy fiber of $\mathbb{S} \to H$. We can inductively construct an Adams resolution

$$\begin{array}{ccccc} X_0 = \mathbb{S} & \longleftarrow & X_1 = \bar{H} & \longleftarrow & \cdots \\ \downarrow & & \downarrow & & \\ W_0 = H \wedge X_0 = H & & W_1 = H \wedge X_1 & & \end{array}$$

Here $X_s = \bar{H}^{\wedge s}$, $W_s = H \wedge X_s = H \wedge \bar{H}^{\wedge s}$.
The fiber sequence $X_{s+1} \to X_s \to W_s$ is induced by smashing $\bar{H} \to \mathbb{S} \to H$ with X_s.

Motivic Adams spectral sequence

This gives the motivic Adams spectral sequence and its E_2-term is

$$Ext_{\mathcal{A}_{*,*}}^{s,(t+s,u)}(\mathbb{F}_p[\tau], \mathbb{F}_p[\tau])$$

Here s is the homological degree, and $(t+s, u)$ is the internal bidegree. Recall that $t+s$ corresponds to the usual topological grading in the classical Steenrod algebra, and u is the motivic weight. Sometimes we also refer to t as the stem and s as the Adams filtration. Motivic Adams spectral sequence converges strongly to the bigraded homotopy groups $\pi_{t,u}(\mathbb{S}_H^{\wedge})$, where $\mathbb{S}_H^{\wedge}$ is the H completion of $\mathbb{S}$.

Motivic Adams spectral sequence

Applying the topological realization functor to our Adams resolution, we obtain the classical Adams resolution. Topological realization gives natural maps $\pi_{a,b}(X) \to \pi_a(X(\mathbb{C}))$ for any motivic spectrum X. This gives a map of spectral sequences $E_r^{s,(t+s,u)} \to E_r^{s,t+s}$. Now we have commutative squares

$$\begin{array}{ccc} E_r^{s,(t+s,u)} & \longrightarrow & E_r^{s,t+s} \\ \downarrow d_r & & \downarrow d_r \\ E_r^{s+r,(t+s+r-1,u)} & \longrightarrow & E_r^{s+r,t+s+r-1} \end{array}$$

Motivic Adams spectral sequence

There are maps of spectral sequences: