An Introduction to Motivic Homotopy Theory part 4: unstable motivic homotopy theory

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June 4, 2021

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The output of this machinery is the unstable motivic category.

Step 0: category of schemes

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We say X is *smooth* if each point of X has an open neighborhood which is a smooth affine scheme of some dimension over k. Smooth schemes play the role in algebraic geometry of manifolds in topology.
We let Sm/k denote the category of smooth schemes of finite type over k. Our goal is to set up a homotopy theory for these schemes.

Step 1: embed into category of presheaves

The first problem coming across is that the category Sm/k is far from being complete and cocomplete in the categorical sense. To solve the problem, we need to add in all necessary limits and colimits. There is a general method for this.

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Let C be a small category. There is a canonical functor

$$r : C \to Funct(C^{op}, Set)$$

called the Yoneda embedding, which sends any object $A \in C$ to the representable presheaf $rA \in Funct(C^{op}, Set)$ defined as

$$Hom_C(-, A) : C^{op} \to Set$$

$$D \mapsto Hom_C(D, A)$$

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As in any category of functors, the (co)limits in $Funct(C^{op}, Set)$ can be computed object-wise. This means that $Funct(C^{op}, Set)$ is both complete and cocomplete, since Set is. Moreover, this free cocompletion construction is universal:

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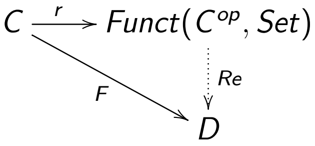
Any functor $F : C \to D$ where D is cocomplete can be factored uniquely up to unique isomorphism through a colimit-preserving functor, as in the diagram



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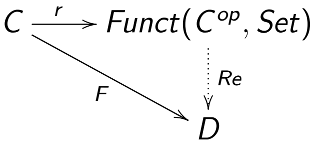

In our case, take $C = Sm/k$, our first step is to embed Sm/k into the category of presheaves $Funct((Sm/k)^{op}, Set)$.

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Hence we further embed $Funct((Sm/k)^{op}, Set)$ into the category of simplicial presheaves $Funct((Sm/k)^{op}, sSet)$ by regarding sets as constant simplicial sets.

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As any category of functors, $Funct((Sm/k)^{op}, sSet)$ inherits most of the structure of the target category $sSet$.

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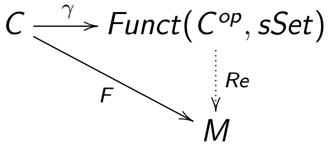

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$$\begin{array}{ccc} C & \xrightarrow{\gamma} & Funct(C^{op}, sSet) \\ & \underset{F}{\searrow} & \Big\downarrow{\scriptstyle Re} \\ & & M \end{array}$$

To simplify notation, we denote $Spc(k) := Funct((Sm/k)^{op}, sSet)$ and call it the category of k-spaces.

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There is a formal process for add weak equivalences called Bousfield localization.

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An object $K \in M$ is said to be *S-local* if it is fibrant in M and if for any morphism $A \xrightarrow{f} B \in S$, the induced map of mapping spaces

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In particular, weak equivalences in M and maps in S are all S-local equivalences.

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This definition does not guarantee that $L_S M$ exists. Even though we can define the three classes of morphisms in M, they may not satisfy the five axioms of a model category.

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- Any isomorphism $U \xrightarrow{\cong} X$ gives a covering family of X with one morphism $\{U \to X\}$

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- For any covering family $\{U_\alpha \to X\}_\alpha$ of X and any map $Y \to X$, the projections $U_\alpha \times_X Y \to Y$ from the pullback squares

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A category C with the additional structure of a Grothendieck topology is called a *site*.

Grothendieck topology

The prototype example of a small site is the category $top(X)$ for a topological space X. The objects of $top(X)$ are the open subsets inclusions $U \to X$. The morphisms are the open subsets inclusions $U \to V$ such that the triangle commutes

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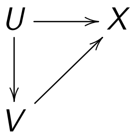

A family of morphisms $\{U_\alpha \to W\}_\alpha$ in the category $top(X)$ is a covering family if and only if $\bigcup U_\alpha$ covers W.

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There are several choices for Grothendieck topology on the category Sm/k. It turns out that the Nisnevich topology is most suitable for motivic homotopy theory. We will work with the Grothendieck topology on Sm/k generated by Nisnevich coverings.

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We say a map $f : X \to Y$ in $Spc(k) := Funct((Sm/k)^{op}, sSet)$ is a *local weak equivalence* if f induces weak equivalences of simplicial sets in all stalks. We let L denote the collection of all local weak equivalences.

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Note here the definition of stalks depend on the Grothendieck topology we choose. The collection L encodes the geometric information of schemes we want to keep.

Based on work of J. Jardine, B. Blander proved the Bousfield localization of the global projective model structure on $Spc(k)$ with respect to L exists. We will call the new model structure the *local projective model structure* on $Spc(k)$.

Step 4: localize to have correct homotopy

The local projective model structure on $Spc(k)$ already has the correct geometric behavior. However, in the homotopy theory we have in mind, the affine line $\mathbb{A}^1_k$ should be our analog of unit interval. In particular, it should be contractible.

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Our final step is to Bousfield localize the local projective model structure on $Spc(k)$ with respect to the map $\gamma\mathbb{A}^1_k \to *$. The new model structure exists, we call it the motivic model structure on $Spc(k)$.

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The associated homotopy category is denoted $H(k)$, and this is the unstable motivic homotopy category.

Two types of spheres

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Hence the category $Spc(k)$ of k-spaces contains copies of both Sm/k and $sSet$. We let $S^{1,1}$ denote the k-space corresponding to the scheme $\mathbb{A}^1_k - 0$. Let $S^{1,0}$ denote the k-space corresponding to the simplicial set S^1.