An Introduction to Motivic Homotopy Theory part 3: scheme

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Introduction

The purpose of this talk is to explain the notion of schemes, which has now become a central object of study in algebraic geometry. Schemes are locally ringed spaces built out of spectra of rings just like manifolds are topological spaces built out of $\mathbb{R}^n$.
For this, we first explain what is a locally ringed space. This requires the idea of sheaves and local rings.

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A *presheaf of sets* on X is a functor $\mathcal{F} : Op(X)^{op} \to Set$. In other words, $\mathcal{F}$ assigns to each open subset $U \subset X$ a set $\mathcal{F}(U)$ and to each inclusion $V \subset U$ a map $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$, called the restriction map, such that $\rho_{UU} = id_{\mathcal{F}(U)}$ and whenever $W \subset V \subset U$ we have $\rho_{UW} = \rho_{VW}\rho_{UV}$. We will use the notation $s|_V := \rho_{UV}(s)$ if $s \in \mathcal{F}(U)$.

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Indeed, consider the rule $\mathcal{F}$ which associates to the open subset $U \subset X$ the set $\mathcal{F}(U) = \{f : U \to \mathbb{R}|$ f is continutous$\}$ with the obvious restriction mappings. Then $\mathcal{F}$ is a presheaf.

Sheaves

A *sheaf of sets* on X is a presheaf of sets $\mathcal{F}$ which satisfies the following "local-to-global" property:
Given any open covering $U = \bigcup_{i \in I} U_i$ and any collection of elements $s_i \in \mathcal{F}(U_i)$, $i \in I$ such that $\forall i, j \in I$

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$$

there exists a unique $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all $i \in I$.

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The presheaf $\mathcal{F}$ of continuous real valued functions is also a sheaf. To see this, suppose that $U = \bigcup_{i \in I} U_i$ is an open covering, and $f_i \in \mathcal{F}(U_i)$, $i \in I$ with $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all $i, j \in I$. In this case define $f : U \to \mathbb{R}$ by setting $f(u)$ equal to the value of $f_i(u)$ for any $i \in I$ such that $u \in U_i$. This is well defined by assumption. Moreover, $f : U \to \mathbb{R}$ is a map such that its restriction to U_i agrees with the continuous map f_i. Hence clearly f is continuous.

Stalks

Let $x \in X$ be a point. Let $\mathcal{F}$ be a presheaf of sets on X. The *stalk* of $\mathcal{F}$ at x is the set $\mathcal{F}_x = colim_{x \in U} \mathcal{F}(U)$ where the colimit is over the set of open neighbourhoods U of x in X, partially ordered by reverse inclusion.

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Note that the colimit is a directed colimit. Thus

$$\mathcal{F}_x = \{(U, s) | x \in U, s \in \mathcal{F}(U)\} / \sim$$

with equivalence relation given by $(U, s) \sim (U', s')$ if and only if there exists an open subset $U'' \subset U \cap U'$ with $x \in U''$ and $s|_{U''} = s'|_{U''}$.

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Again, consider the presheaf of continuous functions on X. A pair of functions $f : U \to \mathbb{R}, g : U' \to \mathbb{R}$ determine the same element of the stalk if there exists a neighbourhood U'' of x such that f and g agree on U''. This corresponds to the classical notion of germs.

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Let $S \subset R$ be a multiplicatively closed subset. Then

$$(r, s) \sim (r', s') \text{ if there exists } u \in S \text{ such that } u(rs' - r's) = 0$$

is an equivalence relation on $R \times S$. We denote the equivalence class of a pair $(r, s) \in R \times S$ by $\frac{r}{s}$.

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The set of all equivalence classes

$$\{\frac{r}{s} | r \in R, s \in S\} =: S^{-1}R$$

is called the *localization of R at S*. It is a ring together with the addition and multiplication

$$\frac{r}{s} + \frac{r'}{s'} := \frac{rs' + r's}{ss'}, \ \frac{r}{s} \cdot \frac{r'}{s'} := \frac{rr'}{ss'}$$

localization of rings

For example, let f be an element in R. Let $S = \{f^n \mid n \geq 0\}$. Then S is a multiplicatively closed subset of R. The corresponding localization $S^{-1}R$ is often written as R_f; we call it the localization of R at the element f.

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As another example, let p be a prime ideal of R. Let $S = R - p$, then S is multiplicatively closed, since $a \notin p$ and $b \notin p$ implies $ab \notin p$. The resulting localization $S^{-1}R$ is usually denoted by R_p and called the localization of R at the prime ideal p.

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Now, we are ready to define locally ringed spaces.
A *locally ringed space* $(X, \mathcal{O}_X)$ is a pair consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$ on X where all of whose stalks are local rings.

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Typical examples of locally ringed spaces are spectra of commutative rings.

Spectrum of a ring - underlying set

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If $\phi : R \to R'$ is a ring homomorphism, and p' is a prime ideal of R', then $\phi^{-1}(p')$ is a prime ideal of R. Therefore, there's induced map $Spec(\phi) : Spec(R') \to Spec(R)$, $p' \to \phi^{-1} p'$.

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Again, we look at the example where $R = \mathbb{C}[x]$. $f(x) = x^2 - 3x + 1$ is an element of R. It defines a function on $Spec(R)$, where at $(x - a)$, the function value is f mod $(x - a) = f(a)$. The value of f at (0) is f(x) (mod 0), which is just f(x).

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Each element $r \in R$ can be viewed as a function on $Spec(R)$, the value of r on $p \in Spec(R)$ is $r \bmod p$ in R/p.
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To define a topology on $Spec(R)$, the intuition is that the subset of $Spec(R)$ where a function vanishes should reasonably be a closed set, and the Zariski topology is defined by saying that the only sets we should consider closed should be these sets, and other sets forced to be closed by these.

Spectrum of a ring - topology

If S is a subset of a ring R, define the Vanishing set of S by

$$V(S) = \{p \in Spec(R) | S \subset p\}$$

It is the set of points on which all elements of S evaluate to zero. (Recall that "vanishing at a point" means the same thing as "contained in a prime".)

The *Zariski topology* on $Spec(R)$ is defined by declaring that these — and no other — are the closed subsets of $Spec(R)$.

Spectrum of a ring - topology

Again, we examine the Zariski topology on $Spec(\mathbb{C}[x])$ to illustrate the idea.

We can associate the point $(x-a)$ in $Spec(\mathbb{C}[x])$ to $a \in \mathbb{C}$. Besides the "traditional" points for each complex number, there is also an extra point (0). The closed sets in $Spec(\mathbb{C}[x])$ include the entire space, and the union of a finite number of "traditional" points. Note the new point (0) is not closed. If a closed set contains (0), then it must be $V(\{0\})$, which is the entire space.

Therefore, we can mostly picture $Spec(\mathbb{C}[x])$ as $\mathbb{C}$. For the extra point (0), we will somehow associate it with the complex plane passing through all the other points.

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Given a subset $S \subset R$, the complement of $V(S)$ is just $\cup_{f \in S} D(f)$. Hence the distinguished open sets form a basis of open sets for the Zariski topology on Spec(R).

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Suppose that $\phi : R \to R'$ is a ring homomorphism. The induced map $Spec(\phi) : Spec(R') \to Spec(R)$, $p' \to \phi^{-1}p'$ is continuous for the Zariski topologies. In fact, for any element $f \in R$ we have $Spec(\phi)^{-1}(D(f)) = D(\phi(f))$. Therefore, Spec defines a functor $Spec : Ring^{op} \to Top$.

Spectrum of a ring - structure sheaf

Lemma: If $f, g \in R$, then $D(f) \cap D(g) = D(fg)$.
Lemma: Let $f, g \in R$ such that $D(g) \subset D(f)$, then $g^e = af$ for some $e \geq 1, a \in R$. Hence, there is a canonical ring map $R_f \to R_g$

$$\frac{b}{f^n} \to \frac{a^n b}{g^{ne}}$$

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Let R be a commutative ring. The structure sheaf $\mathcal{O}_{Spec(R)}$ on $Spec(R)$ is defined as the unique sheaf of rings such that:
(a) on the standard opens $\mathcal{O}_{Spec(R)}(D(f)) = R_f$.
(b) If $D(g) \subset D(f)$, then the restriction map

$$\mathcal{O}_{Spec(R)}(D(f)) \to \mathcal{O}_{Spec(R)}(D(g))$$

is the canonical ring map $R_f \to R_g$.

Spectrum of a ring - structure sheaf

From the definition, on the open set $Spec(R) = D(u)$ for unit $u \in R$,

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Let x be a point in $Spec(R)$ corresponding to a prime ideal $p \subset R$. Then the stalk of $\mathcal{O}_{Spec(R)}$ at x can be computed as

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Therefore $(Spec(R), \mathcal{O}_{Spec(R)})$ is a locally ringed space. We call it the *spectrum of R* and still denote it by $Spec(R)$.
As a convention, we denote $\mathbb{A}^1_{\mathbb{C}} := Spec(\mathbb{C}[x])$ and

$$\mathbb{A}^n_{\mathbb{C}} := Spec(\mathbb{C}[x_1, x_2, \cdots, x_n])$$

Schemes

An affine scheme is a locally ringed space isomorphic as a locally ringed space to Spec(R) for some commutative ring R.
A scheme is a locally ringed space with the property that every point has an open neighbourhood which is an affine scheme.
One can show that an open subspace U of a scheme X is still a scheme (with the restriction of $\mathcal{O}_X$ to U as structural sheaf), but an open subspace of an affine scheme is not necessarily affine.

Schemes

Consider $\mathbb{A}^1_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x]$. Let $U = D(x)$ be an open subspace of $\mathbb{A}^1_{\mathbb{C}}$, we have $\mathcal{O}_{\mathbb{A}^1_{\mathbb{C}}}(U) = \mathbb{C}[x]_x = \mathbb{C}[x, 1/x]$. Moreover, $\mathcal{O}_{\mathbb{A}^1_{\mathbb{C}}}$ restricts to a sheaf on U, making U a locally ringed space.

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One can show $U = Spec(\mathbb{C}[x, 1/x])$. Intuitively, the reason is as follows: given an element f of a ring R, there is a bijection of prime ideals of R_f and the prime ideals of R not containing f. Hence when we study prime ideals of $\mathbb{C}[x]_x$, we only need to take the results for $\mathbb{C}[x]$, then restrict to $D(x)$.

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Since we can intuitively think of the point $(x) \in \mathbb{A}^1_{\mathbb{C}}$ as $0 \in \mathbb{C}$, we usually denote the scheme U as $\mathbb{A}^1_{\mathbb{C}} - 0$. In the following talks, we will see $\mathbb{A}^1_{\mathbb{C}} - 0$ corresponds to the motivic sphere $S^{1,1}$.

What's next

Next time, we will discuss the unstable motivic homotopy theory, which can be described as the associated homotopy category of a model structure on $Fun((Sm/k)^{op}, sSet)$. We will talk about the model structure, as well as some motivation behind such constructions.