An Introduction to Motivic Homotopy Theory part 2: model category

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Pushout in general categories

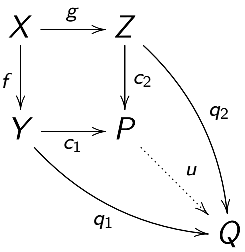
Let C be a category. Let $f : X \to Y$ and $g : X \to Z$ be two morphisms in C. A *pushout* of the diagram $Y \xleftarrow{f} X \xrightarrow{g} Z$ in C is an object P together with two morphisms $c_1 : Y \to P$ and $c_2 : Z \to P$ for which the diagram

$$\begin{array}{ccc} X & \xrightarrow{g} & Z \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle c_2} \\ Y & \xrightarrow[c_1]{} & P \end{array}$$

commutes. Moreover, the pullback must be universal in the following sense.

Pushout in general categories

For any other such triple (Q, q_1, q_2) where $q_1 : Y \to Q$ and $q_2 : Z \to Q$ are morphisms with $q_1 f = q_2 g$, there exists a unique $u : P \to Q$ such that the following diagram commutes.



If a pullback exists in C, then it is unique up to canonical isomorphism.

Pushout in general categories

In Set, the pushout of $Y \xleftarrow{f} X \xrightarrow{g} Z$ always exists and is given by

$$\begin{array}{ccc} X & \xrightarrow{g} & Z \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle c_2} \\ Y & \xrightarrow[c_1]{} & P \end{array}$$

$P = (Y \coprod Z)/\sim$, where $\sim$ is the equivalence relation generated by requiring $fx \sim gx$ for all $x \in X$.
If f, g are both inclusion of subsets, then $P = Y \cup Z$.

Pushout of topological spaces

In sSet, the pushout P of $Y \xleftarrow{f} X \xrightarrow{g} Z$ always exists and is computed levelwise. In other words, $P([k])$ is the pushout of sets

$$Y([k]) \xleftarrow{f_k} X([k]) \xrightarrow{g_k} Z([k])$$

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In Top, the pushout P of $Y \xleftarrow{f} X \xrightarrow{g} Z$ always exists and is given by $P = (Y \coprod Z)/\sim$, where $\sim$ is the equivalence relation generated by requiring $fx \sim gx$ for all $x \in X$.

Pushout of topological spaces

There are pushout squares

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{i} & D^n \\ {\scriptstyle i}\downarrow & & \downarrow \\ D^n & \longrightarrow & S^n \end{array} \qquad \begin{array}{ccc} S^{n-1} & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & * \end{array}$$

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We have a commutative diagram

$$\begin{array}{ccccc} D^n & \xleftarrow{i} & S^{n-1} & \xrightarrow{i} & D^n \\ \downarrow\wr & & \| \wr & & \downarrow\wr \\ * & \longleftarrow & S^{n-1} & \longrightarrow & * \end{array}$$

where the vertical arrows are homotopy equivalences. However, the induced map of pushouts $S^n \to *$ is not a homotopy equivalence.

Pushout of topological spaces

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The problem is Ho(Top) does not have all pushouts. For example, one can show the push out of $* \leftarrow S^1 \xrightarrow{\times 2} S^1$ does not exist in Ho(Top).
A better solution is provided by using a model structure on Top.

Model category

A *model category* is a category C with three distinguished classes of maps: (i) weak equivalences, (ii) fibrations, and (iii) cofibrations each of which is closed under composition and contains all identity maps. A map which is both a fibration (resp. cofibration) and a weak equivalence is called an acyclic fibration (resp. acyclic cofibration). We require the following axioms:

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We require the following axioms:

Axiom 1: Limits and colimits exist in C.

Axiom 2: If two of f, g, gf are weak equivalences, so is the third.

Axiom 3: If f is a retract of g and g is a fibration, cofibration, or a weak equivalence, then so is f.

$$\begin{array}{ccccc} X & \xrightarrow{i} & A & \xrightarrow{j} & X \\ \downarrow f & & \downarrow g & & \downarrow f \\ Y & \xrightarrow{h} & B & \xrightarrow{k} & Y \end{array} \qquad \begin{array}{c} ji = Id_X \\ \\ kh = Id_Y \end{array}$$

Model category

Axiom 4: Given a commutative diagram of the form

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

a lift h exists in either of the following two situations:
(i) i is a cofibration and p is an acyclic fibration, or
(ii) i is an acyclic cofibration and p is a fibration.

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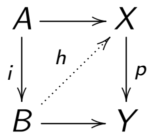
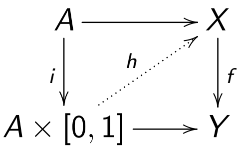


a lift h exists in either of the following two situations:
(i) i is a cofibration and p is an acyclic fibration, or
(ii) i is an acyclic cofibration and p is a fibration.
Axiom 5: Any map f can be functorially factored in two ways:
(i) f = pi, where i is a cofibration and p is an acyclic fibration, and
(ii) f = pi, where i is an acyclic cofibration and p is a fibration.

Model category

The category Top of topological spaces can be given the structure of a model category by defining $f : X \to Y$ to be

- a weak equivalence if f is a weak homotopy equivalence
- a cofibration if f is a retract of a map $X \to Y'$ in which Y' is obtained from X by attaching cells
- a fibration if f is a Serre fibration, i.e. if f has homotopy lifting property for each CW-complex A



(Co)fibrant objects

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A *terminal object* in a category C is an object $*$ such that for any object X of C, there is a unique morphism $X \to *$. A terminal object, if it exists, is unique up to unique isomorphism.
A model category C has both an initial object $\emptyset$ and a terminal object $*$ (by Axiom 1). An object $X \in C$ is said to be *cofibrant* if $\emptyset \to X$ is a cofibration, and *fibrant* if $X \to *$ is a fibration.

(Co)fibrant objects

For each object X in C we can apply axiom 5 to the map $\emptyset \to X$ and obtain a natural acyclic fibration

$$p_X : QX \xrightarrow{\sim} X$$

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For each object X in C we can apply axiom 5 to the map $\emptyset \to X$ and obtain a natural acyclic fibration

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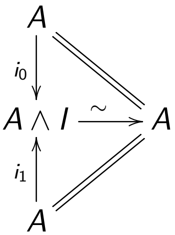
We can also apply the axiom to the map $X \to *$ and obtain a natural acyclic cofibration

$$i_X : X \overset{\sim}{\hookrightarrow} RX$$

with RX fibrant. This suggests we have natural ways to replace general objects by weakly equivalent better behaved objects.

Homotopy relation

Let C be a model category, A be an object in C. A *cylinder object* for A is an object $A \wedge I$ of C together with a commutative diagram



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$$\begin{array}{ccc} A & & \\ {\scriptstyle i_0}\downarrow & \searrow{=} & \\ A \wedge I & \xrightarrow{\sim} & A \\ {\scriptstyle i_1}\uparrow & \nearrow{=} & \\ A & & \end{array}$$

The notation $A \wedge I$ is meant to suggest the product of A with an interval, however, a cylinder object $A \wedge I$ is not necessarily the product of A with anything in C. An object A of C might have many cylinder objects associated to it. We do not assume that there is some preferred natural cylinder object for A.

Homotopy relation

Two maps $f, g : A \to X$ in C are said to be *homotopic* (written $f \sim g$) if there exists a cylinder object $A \wedge I$ for A together with a map $H : A \wedge I \to X$ making the following diagram commutes

$$
\begin{array}{ccc}
A & & \\
{\scriptstyle i_0}\downarrow & \searrow^{f} & \\
A \wedge I & \xrightarrow{H} & X \\
{\scriptstyle i_1}\uparrow & \nearrow_{g} & \\
A & &
\end{array}
$$

Such a map H is said to be a homotopy from f to g.

Homotopy relation

If A and X are both fibrant and cofibrant, then

- $\sim$ is an equivalence relation on $Hom_C(A, X)$. We let $\pi(A, X)$ denote the set of equivalence classes of $Hom_C(A, X)$.

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- A map $f : A \to X$ is a weak equivalence if and only if f has a homotopy inverse, i.e., if and only if there exists a map $g : X \to A$ such that the composites gf and fg are homotopic to the respective identity maps.

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The adjunction

$$sSet \underset{Sing}{\overset{|\ |}{\rightleftarrows}} Top$$

induce equivalence of homotopy categories

$$Ho(sSet) \rightleftarrows Ho(Top)$$

Derived functors

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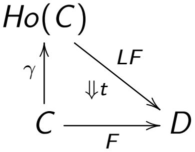
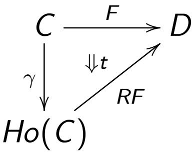
Let C be a model category, $F : C \to D$ be a functor. Next we define the best possible approximations to an "extension of F to Ho(C)". Suppose that F(f) is an isomorphism whenever f is a weak equivalence between cofibrant objects in C. Then the functor $LF := FQ$ is called the *left derived functor* of F, where Q is the cofibrant replacement functor. There exists a diagram



making LF "universal from the left".

Derived functors

A *right derived functor* for F is a functor $RF : Ho(C) \to D$ constructed similarly with the analogous property of being "universal from the right".



Homotopy pushout

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Let D denote the category $\{a \leftarrow b \to c\}$ and Top^D denote the category of functors $D \to Top$. An object of Top^D is pushout data $X(a) \leftarrow X(b) \to X(c)$ in Top and a morphism $f : X \to Y$ is a commutative diagram

$$\begin{array}{ccccc} X(a) & \longleftarrow & X(b) & \longrightarrow & X(c) \\ \downarrow f_a & & \downarrow f_b & & \downarrow f_c \\ Y(a) & \longleftarrow & Y(b) & \longrightarrow & Y(c) \end{array}$$

The pushout construction gives a functor $P : Top^D \to Top$.

Homotopy pushout

The category Top^D has a model structure where a map $f : X \to Y$ in Top^D is

- a weak equivalence if the morphisms f_a, f_b, f_c are weak equivalences in Top
- a fibration if the morphisms f_a, f_b, f_c are fibrations in Top

In this model structure, an object $X(a) \leftarrow X(b) \to X(c)$ is cofibrant if and only if $X(a), X(b), X(c)$ are cofibrant in Top and the maps $X(b) \to X(a), X(b) \to X(c)$ are cofibrations in Top.

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However, one can show there is a (total) left derived functor LP : $Ho(Top^D) \to Ho(Top)$ which in a certain sense is the best possible homotopy invariant approximation to pushout. The resulting functor LP is called homotopy pushout.

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In practice, LP(X) is computed as P(X') for any cofibrant object X' of Top^D weakly equivalent to X.

$$LP(* \leftarrow S^{n-1} \to *) = P(D^n \leftarrow S^{n-1} \to D^n) = S^n$$

Tor and Ext

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Hence Tor and Ext are derived functors of $\otimes$ and *Hom*. In other words, Tor and Ext are homotopical corrections of the original functors.

What's next

The (unstable) motivic homotopy category can be viewed as the homotopy category of a model category $Fun((Sm/k)^{op}, sSet)$, where Sm/k is a category of schemes. We have talked about simplicial sets in the first talk. We will introduce the notion of schemes next time.