An Introduction to Motivic Homotopy Theory part 1: overview and category theory

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Overview

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There are functors between different categories as follows:

$$\begin{array}{ccc} SH(\mathbb{R}) & \xrightarrow{R_{\mathbb{C}}} & SH^{C_2} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle U} \\ SH(\mathbb{C}) & \xrightarrow{R_{\mathbb{C}}} & SH \end{array}$$

$SH(\mathbb{R})$ is the $\mathbb{R}$-motivic stable homotopy category
$SH(\mathbb{C})$ is the $\mathbb{C}$-motivic stable homotopy category
SH^{C_2} is the C_2-equivariant stable homotopy category
SH is the classical stable homotopy category
$R_{\mathbb{C}}$ is the realization functor defined by taking $\mathbb{C}$-points of a scheme
U is the forgetful functor
F is the base change functor

Classical homotopy and motivic homotopy

classical stable homotopy	$\mathbb{C}$-motivic stable homotopy
S^1	$S^{1,0}, S^{1,1}$
$S^n = (S^1)^{\wedge n}$	$S^{n,m} = (S^{1,0})^{\wedge n-m} \wedge (S^{1,1})^{\wedge m}$
π_*	$\pi_{*,*}$
H_*, H^*	$H_{*,*}, H^{*,*}$
$\mathcal{A}^*, \mathcal{A}_*$	$\mathcal{A}^{*,*}, \mathcal{A}_{*,*}$
Adams spectral sequence	Motivic Adams spectral sequence

$\Longleftarrow$ realization

Classical homotopy and motivic homotopy

There are maps of spectral sequences:

motivic Adams for $\hat{S}^{0,0}$

$\swarrow$ $\searrow$

motivic Adams for $\hat{S}^{0,0}/\tau$ Adams for $\hat{S}^{0}$

$\|$

classical algebraic Novikov

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3. Composition is associative, that is: given $f : X \to Y$, $g : Y \to Z$ and $h : Z \to W$, $h(gf) = (hg)f$
4. For every object X there is an identity arrow $id_X : X \to X$, satisfying $id_X g = g$ for every $g : Y \to X$ and $f id_X = f$ for every $f : X \to Y$

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We give several examples of categories:
1. *Set* is the category which has the class of all sets as objects, and functions between sets as morphisms.
2. *Ab* is the category of abelian groups and group homomorphisms.
3. *Top* is the category of topological spaces and continuous functions.
4. Given a category C we can form its opposite category C^{op} which has the same objects and morphisms as C, but with reversed direction; so if $f : X \to Y$ in C then $f : Y \to X$ in C^{op}.

Functor

Let C and D be categories. A **functor** F from C to D is a mapping that associates each object X in C to an object $F(X)$ in D, associates each morphism $f : X \to Y$ in C to a morphism $F(f) : F(X) \to F(Y)$ in D such that: $F(id_X) = id_{F(X)}$ and $F(gf) = F(g)F(f)$.
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There is also a "free" functor $Free : Set \to Ab$ which assigns to any set X the free abelian group generated by the elements in X.

Natural transformation

Let F, G be functors $F, G : C \to D$. A **natural transformation** $\eta : F \to G$ associates, to every object X in C, a morphism $\eta_X : F(X) \to G(X)$. For every $f : X \to Y$ in C, the assignment η is required to make the following diagram commutes in D:

$$\begin{array}{ccc} F(X) & \xrightarrow{\eta_X} & G(X) \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ F(Y) & \xrightarrow{\eta_Y} & G(Y) \end{array}$$

Adjoint functors

Let $F : C \to D$, $G : D \to C$ be a pair of functors such that for all objects $X \in C$, $Y \in D$, there is a natural bijection of morphism sets

$$Hom_D(FX, Y) \cong Hom_C(X, GY)$$

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Then we say F is left adjoint to G, G is right adjoint to F.
For example, the free functor $Free : Set \to Ab$ is left adjoint to the forgetful functor $U : Ab \to Set$: we have natural bijections

$$Hom_{Ab}(Free(X), G) \cong Hom_{Set}(X, U(G))$$

This is because a map between free abelian groups is determined by where it sends the generators, and a generator can be mapped to anything.

Functor category

Suppose C is a small category and D is an arbitrary category. There is a category of functors from C to D, written as $Fun(C, D)$ or D^C. The objects are the functors from C to D, and the morphisms are the natural transformations between such functors.

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Functor category D^C can inherit many nice properties of D.
Spoiler: the unstable motivic homotopy category is a functor category $Fun((Sm/k)^{op}, sSet)$, where Sm/k is a category of schemes, $sSet$ is the category of simplicial sets.

Simplicial sets

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The family of sets SX_n together with the structure maps form a typical example of simplicial sets.

Simplicial sets

The simplex category Δ: the objects are non-empty finite totally ordered sets $[n] = \{0, 1, \ldots, n\}$, $n \geq 0$, and the morphisms are weakly order-preserving functions ($f(i) \geq f(j)$ if $i > j$).

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For each $n \geq 0$, there is a simplicial set $\Delta[n] : \Delta^{op} \to Set$ defined as $Hom_\Delta(-, [n])$. In other words, $\Delta[n]([m]) = Hom_\Delta([m], [n])$, the set of all weakly order-preserving functions from $[m]$ to $[n]$.

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Intuitively, the realization of X is the topological space obtained if every n-simplex of X is replaced by a topological n-simplex and these topological simplices are glued together in the fashion the simplices of X hang together.

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There is an adjunction of categories

$$sSet \underset{Sing}{\overset{|\ |}{\rightleftarrows}} Top$$

Convenient category for doing homotopy

Suppose now we want to find a categorical setting for doing topology. A natural choice is Top, where the objects are all topological spaces and the morphisms are all continuous maps.
However, from a categorical point of view, this category does not has all the nice properties we want.

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For example, $Map(\mathbb{N}, [0,1]) = \prod_{n\in\mathbb{N}}[0,1]$ is not a CW complex. Note a CW complex need to be the countable union of its finite dimensional skeletons.

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Moreover, we also have homeomorphisms

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sSet is also cartesian closed. This means there are bijections of Hom sets

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