THE p-PRIMARY SUBGROUP OF THE COHOMOLOGY OF BPU_n IN DIMENSION $2p+6$

YU ZHANG, ZHILEI ZHANG, AND LINAN ZHONG*

Abstract. Let PU_n denote the projective unitary group of rank n, and let BPU_n be its classifying space. We show that the p-primary subgroup of $H^{2p+6}(BPU_n;\mathbb{Z})$ is trivial, where p is an odd prime.

1. Introduction

The purpose of this brief paper is to examine the integral cohomology of BPU_n. The notation U_n refers to the group of $n \times n$ unitary matrices. The projective unitary group, denoted by PU_n, is defined as the quotient group of U_n by S^1, with the identification of S^1 as the normal subgroup of scalar matrices of U_n. Lastly, BPU_n denotes the classifying space of PU_n.

The cohomology of BPU_n is a fundamental object in algebraic topology with broad relevance. It plays a significant role in the study of the period-index problem in algebraic geometry and algebraic topology, as highlighted in works such as [1], [2], [7], and [8]. Additionally, it is crucial in the exploration of anomalies in particle physics, as evidenced by works like [4] and [6]. Other related works include [5], which provides a complete determination of the integral cohomology of PU_n.

The cohomology of BPU_n for special values of n has been extensively investigated by various researchers, including Kameko-Yagita [11], Kono-Mimura [12], Kono-Yagita [13], Toda [15], and Vavpetič-Viruel [16]. Among these works, the only case that has been well-understood so far is when $n = p$, where p is a prime number.

On the other hand, very little is known about the cohomology of BPU_n for arbitrary n, as it is widely regarded as a highly challenging problem. Indeed, none of the aforementioned works have delved into $H^*(BPU_n;\mathbb{Z})$, the ordinary cohomology of BPU_n with coefficients in $\mathbb{Z}$, for non-prime numbers n. However, a recent breakthrough in this field has been achieved by Gu in [9], where the ring structure of $H^*(BPU_n;\mathbb{Z})$ in dimensions less than or equal to 10 for any value of n is determined.

Before delving into further computational results, we would like to introduce some notations that will be utilized throughout this paper.

Notations 1.1. *To simplify notations, we let $H^*(-)$ denote the integral cohomology $H^*(-;\mathbb{Z})$. Given an abelian group A and a prime number p, we let $A_{(p)}$ denote the localization of A at p, and let ${}_pA$ denote the p-primary subgroup of A. In other*

2020 *Mathematics Subject Classification.* 55T10, 55R35, 55R40.

Key words and phrases. Serre spectral sequences, classifying spaces, projective unitary groups.

Supported by National Natural Science Foundation of China (Grant No. 12001474; 12261091; 12271183).

* Corresponding author.

words, ${}_pA$ is the subgroup of A consisting of all torsion elements whose order is a power of p. One useful observation is that there exists a canonical isomorphism ${}_pH^*(-) \cong {}_p[H^*(-)_{(p)}]$. We will use these two interchangeably. Lastly, note that when we take tensor products of $\mathbb{Z}_{(p)}$-modules, we do so over $\mathbb{Z}_{(p)}$.

In the following discussion, we outline our strategy for studying $H^*(BPU_n)$ for arbitrary n. Firstly, it is worth noting that the torsion-free component of $H^*(BPU_n)$ is already thoroughly understood.

For a fixed positive integer n, there is a short exact sequence of Lie groups

$$1 \to \mathbb{Z}/n \to SU_n \to PSU_n \simeq PU_n \to 1,$$

which induces a fiber sequence of their classifying spaces

$$B(\mathbb{Z}/n) \to BSU_n \to BPU_n \tag{1.1}$$

Recall the cohomology of BSU_n is given by

$$H^*(BSU_n) = \mathbb{Z}[c_2, c_3, \cdots, c_n],\ |c_i| = 2i \tag{1.2}$$

We choose a prime number p which does not divide n, then the space $B(\mathbb{Z}/n)$ is p-locally contractible. From (1.1), we get

$$H^*(BPU_n; \mathbb{Z}_{(p)}) \cong H^*(BSU_n; \mathbb{Z}_{(p)}) \tag{1.3}$$

Since $\mathbb{Z}_{(p)}$ is a flat $\mathbb{Z}$-module, $H^*(-;\mathbb{Z}_{(p)}) \cong H^*(-)_{(p)}$. We have an isomorphism of $\mathbb{Z}_{(p)}$-algebras

$$H^*(BPU_n)_{(p)} \cong H^*(BSU_n)_{(p)} = \mathbb{Z}_{(p)}[c_2, c_3, \cdots, c_n], \quad p \nmid n. \tag{1.4}$$

Hence, we can conclude the rank of the torsion-free part of $H^s(BPU_n)$ is just the number of monomials in $c_2, c_3, \ldots, c_n$ in dimension s.

The remaining task is to determine the torsion part of $H^*(BPU_n)$. Following the standard approach in algebraic topology, we work with one prime at a time. Specifically, for arbitrary prime p we study the p-primary subgroup ${}_pH^*(BPU_n)$. It is important to note that if $p \nmid n$, then $H^*(BPU_n)_{(p)}$ is torsion-free, as shown in (1.4). Thus, we have

$${}_pH^*(BPU_n) = {}_p[H^*(BPU_n)_{(p)}] = 0, \quad p \nmid n. \tag{1.5}$$

The interesting cases occur only when $p \mid n$.

In [10], the work accomplished a comprehensive description of $H^s(BPU_n;\mathbb{Z})_{(p)}$ for $s < 2p+5$ by proving that ${}_pH^s(BPU_n) = 0$ for $s = 2p+3$ and $s = 2p+4$ (when p is an odd prime). In this paper, we extend the findings in [10] by computing ${}_pH^s(BPU_n)$ for $s = 2p+6$ for all n. Our main theorem is as follows.

Theorem 1. *Let $p > 2$ be a prime number, and n be a positive integer. Then the p-primary subgroup of the cohomology of BPU_n is trivial in dimension $2p+6$. In other words, we have ${}_pH^{2p+6}(BPU_n) = 0$.*

Remark 1.2. When $p = 2$, the explicit computation of the cohomology of BPU_n in dimension $2p+6 = 10$ can be found in [9], which reveals that

$${}_2H^{10}(BPU_n) \cong \begin{cases} \mathbb{Z}/2, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

Thus, it is evident that the result in Theorem 1 does not apply when $p = 2$.

The result of ${}_pH^{2p+5}(BPU_n)$ is still an open question. It turns out that the problem in dimension $2p+5$ is more complex than in other dimensions. We will say more about this at the end of this paper.

Organization of the paper. In Section 2, we introduce the Serre spectral sequence that we use to compute the cohomology of BPU_n. Additionally, we recall some fundamental results regarding the differentials in the spectral sequence. In Section 3, we provide explicit computations of all relevant differentials and prove Theorem 1. Furthermore, we will discuss the primary difficulty of computing ${}_pH^{2p+5}(BPU_n)$.

Acknowledgments. The authors would like to thank Xing Gu for the helpful discussions which motivated the current project. The authors were supported by the National Natural Science Foundation of China (No. 12001474; 12261091). The first and second named authors were also supported by the National Natural Science Foundation of China (No. 12271183). All authors contribute equally.

2. The spectral sequences

Our tool to compute the cohomology of BPU_n is the Serre spectral sequence ${}^UE$ described in equation (2.2). The same spectral sequence ${}^UE$ was also essential in the related computations presented in [9, 10]. This section serves to refresh the basic framework and computational outcomes for ${}^UE$.

2.1. The Serre spectral sequence ${}^UE$.

The short exact sequence of Lie groups

$$1 \to S^1 \to U_n \to PU_n \to 1$$

induces a fiber sequence of their classifying spaces

$$BS^1 \to BU_n \to BPU_n$$

Notice that BS^1 has the homotopy type of the Eilenberg-Mac Lane space $K(\mathbb{Z},2)$, there is an associated fiber sequence

$$U: \ BU_n \to BPU_n \to K(\mathbb{Z},3) \tag{2.1}$$

We will use the Serre spectral sequence associated to (2.1) to compute the cohomology of BPU_n. For notational convenience, we denote this spectral sequence by ${}^UE$. The E_2 page of ${}^UE$ has the form

$${}^UE_2^{s,t} = H^s(K(\mathbb{Z},3); H^t(BU_n)) \Longrightarrow H^{s+t}(BPU_n) \tag{2.2}$$

To carry out actual computations with this spectral sequence, we need to know the cohomology of $K(\mathbb{Z},3)$ and BU_n. As we will see in (3.1), since the purpose of this paper is to study the p-primary subgroup of $H^*(BPU_n)$ for a fixed prime p, it suffices to know the p-local cohomology of $K(\mathbb{Z},3)$.

We summarize the p-local cohomology of $K(\mathbb{Z},3)$ in low dimensions as follows. The original reference is [3], also see [14] for a nice treatment.

Proposition 2.2. *Let $p>2$ be a prime. In degrees up to $2p+7$, we have*

$$H^s(K(\mathbb{Z},3))_{(p)} = \begin{cases} \mathbb{Z}_{(p)}, & s=0,\ 3, \\ \mathbb{Z}/p, & s=2p+2,\ 2p+5, \\ 0, & s\leq 2p+7, s\neq 0,\ 3,\ 2p+2,\ 2p+5. \end{cases} \tag{2.3}$$

where $x_1,\ y_{p,0},\ x_1y_{p,0}$ are generators on degree $3, 2p+2, 2p+5$ respectively.

Remark 2.3. Here the notations for the generators are taken from [9, 2.14].

Also recall

$$H^*(BU_n) = \mathbb{Z}[c_1, c_2, \dots, c_n],\ |c_i| = 2i \tag{2.4}$$

In particular, $H^*(BU_n)$ is torsion-free. We have

$${}^UE_2^{s,t} \cong H^s(K(\mathbb{Z},3)) \otimes H^t(BU_n) \tag{2.5}$$

2.4. **The auxiliary spectral sequences ${}^TE$ and ${}^KE$.** Instead of computing the differentials in ${}^UE$ directly, which is difficult in practice, our strategy is to compare ${}^UE$ with two auxiliary spectral sequences, which has simpler differential behaviors. We now introduce the two auxiliary fiber sequences and their associated Serre spectral sequences.

Let T^n be the maximal torus of U_n with the inclusion denoted by

$$\psi : T^n \to U_n.$$

Passing to quotients over S^1, we have another inclusion of maximal torus

$$\psi' : PT^n \to PU_n.$$

The quotient map $T^n \to PT^n$ fits into an exact sequence of Lie groups

$$1 \to S^1 \to T^n \to PT^n \to 1,$$

which induces another fiber sequence of their classifying spaces

$$T:\ BT^n \to BPT^n \to K(\mathbb{Z},3) \tag{2.6}$$

T is our first auxiliary fiber sequence.

We also consider the path fibration for $K(\mathbb{Z},3)$

$$K:\ K(\mathbb{Z},2) \simeq BS^1 \to * \to K(\mathbb{Z},3) \tag{2.7}$$

where $*$ denotes a contractible space. K is our second auxiliary fiber sequence.

These fiber sequences fit into the following homotopy commutative diagram:

$$\begin{tikzcd}
K: \arrow[d, "\Phi"] & BS^1 \arrow[r] \arrow[d, "B\varphi"] & * \arrow[r] \arrow[d] & K(\mathbb{Z},3) \arrow[d, "="] \\
T: \arrow[d, "\Psi"] & BT^n \arrow[r] \arrow[d, "B\psi"] & BPT^n \arrow[r] \arrow[d, "B\psi'"] & K(\mathbb{Z},3) \arrow[d, "="] \\
U: & BU_n \arrow[r] & BPU_n \arrow[r] & K(\mathbb{Z},3)
\end{tikzcd} \tag{2.8}$$

Here, the map $B\varphi : BS^1 \to BT^n$ is induced by the diagonal map $\varphi : S^1 \to T^n$.

We denote the Serre spectral sequences associated to U, T, and K as ${}^UE$, ${}^TE$ and ${}^KE$ respectively. We denote their corresponding differentials by ${}^Ud_*^{*,*}$, ${}^Td_*^{*,*}$, and ${}^Kd_*^{*,*}$ respectively. When the actual meaning is clear from the context, we also simply denote the differentials by $d_*^{*,*}$.

In this paper, we compute differentials in ${}^UE$ by comparing them with the differentials in ${}^TE$ and ${}^KE$. This is possible because: (1) we have explicit formulas for the maps between spectral sequences, and (2) we have a good understanding of the corresponding differentials in ${}^TE$ and ${}^KE$.

We first describe the comparison maps between ${}^UE$, ${}^TE$ and ${}^KE$.

Notice that we have

$$H^*(BT^n) = \mathbb{Z}[v_1, v_2, \dots, v_n],\ |v_i| = 2. \tag{2.9}$$

The induced homomorphism between cohomology rings is as follows:

$$B\varphi^* : H^*(BT^n) = \mathbb{Z}[v_1, v_2, \cdots, v_n] \to H^*(BS^1) = \mathbb{Z}[v],\ v_i \mapsto v.$$

The map $B\psi : BT^n \to BU_n$ induces the injective ring homomorphism

$$\begin{aligned} B\psi^* : H^*(BU_n) = \mathbb{Z}[c_1, \cdots, c_n] \to H^*(BT^n) = \mathbb{Z}[v_1, \cdots, v_n], \\ c_i \mapsto \sigma_i(v_1, \cdots, v_n), \end{aligned} \tag{2.10}$$

where $\sigma_i(t_1, t_2, \cdots, t_n)$ is the ith elementary symmetric polynomial in variables $t_1, t_2, \cdots, t_n$:

$$\begin{aligned} &\sigma_0(t_1, t_2, \cdots, t_n) = 1, \\ &\sigma_1(t_1, t_2, \cdots, t_n) = t_1 + t_2 + \cdots + t_n, \\ &\sigma_2(t_1, t_2, \cdots, t_n) = \sum_{i<j} t_i t_j, \\ &\vdots \\ &\sigma_n(t_1, t_2, \cdots, t_n) = t_1 t_2 \cdots t_n. \end{aligned} \tag{2.11}$$

We also recall some important propositions regarding the higher differentials in ${}^KE$ and ${}^TE$. The following result of differentials in ${}^KE$ is the starting point for relevant computations in ${}^TE$ and ${}^UE$.

Proposition 2.5. *The higher differentials of ${}^KE_*^{*,*}$ satisfy*

$$\begin{aligned} &d_3(v) = x_1, \\ &d_{2p-1}(x_1 v^{lp^e - 1}) = v^{lp^e - 1 - (p-1)} y_{p,0}, \quad e > 0,\ \gcd(l, p) = 1, \\ &d_r(x_1) = d_r(y_{p,0}) = 0, \quad \textit{for all } r, \end{aligned}$$

and the Leibniz rule.

Remark 2.6. Proposition 2.5 is a special case of [9, Corollary 2.16]. Note there is a typo in the original reference, where the condition $k \geq e$ should be replaced by $e > k$.

By comparing with the differentials in ${}^KE$, one could obtain the following results on differentials in ${}^TE$.

Proposition 2.7 ([10], Lemma 3.1)**.** *In the spectral sequence ${}^TE$, we have*

$${}^Td_{2p-1}^{3,*}(v_n^k x_1) = 0$$

for $0 \leq k \leq p-2$ or $k = p$, and

$${}^Td_{2p-1}^{3,*}(v_n^{p-1} x_1) = y_{p,0}$$

Proposition 2.8 ([9], Proposition 3.3)**.** (1) *The differential ${}^Td_3^{0,t}$ is given by the "formal divergence"*

$$\nabla = \sum_{i=1}^n (\partial/\partial v_i) : H^t(BT^n; R) \to H^{t-2}(BT^n; R),$$

in such a way that ${}^Td_3^{0,*} = \nabla(-) \cdot x_1$. For any ground ring $R = \mathbb{Z}$ or $\mathbb{Z}/m$ for any integer m.

(2) *The spectral sequence degenerates at* ${}^TE_4^{0,*}$. *Indeed, we have* ${}^TE_\infty^{0,*} = {}^TE_4^{0,*} = \operatorname{Ker} {}^Td_3^{0,*} = \mathbb{Z}[v_1 - v_n, \cdots, v_{n-1} - v_n]$.

The following is a useful corollary.

Corollary 2.9. *We have*

$$ {}^Ud_3^{0,*}(c_k) = \nabla(c_k)x_1 = (n-k+1)c_{k-1}x_1 $$

for $2 \leq k \leq n$, *and*

$$ {}^Ud_3^{0,*}(c_1) = nx_1 $$

Remark 2.10. Corollary 2.9 first appeared in [9, Corollary 3.4]. Here, we write out the result for c_1 separately since c_0 is not defined.

3. Computations in the spectral sequence ${}^UE$

The purpose of this section is to provide explicit computations using the Serre spectral sequence ${}^UE$ in order to prove Theorem 1. Noticing

$$ {}_pH^*(BPU_n) \cong {}_p[H^*(BPU_n)_{(p)}], $$

in order to study the p-primary subgroup of $H^*(BPU_n)$, it suffices to look at the p-localized spectral sequence, where the E_2 page becomes

$$ ({}^UE_2^{s,t})_{(p)} = H^s(K(\mathbb{Z},3))_{(p)} \otimes H^t(BU_n) = H^s(K(\mathbb{Z},3)) \otimes H^t(BU_n)_{(p)}. \tag{3.1} $$

Notations 3.1. *From Equation* (1.5)*, we see that the result in Theorem 1 is trivial when* $p \nmid n$. *Therefore, to prove the theorem, we only need to consider the case when* $p \mid n$. *For the remainder of this paper, we will consider a fixed prime* $p \geq 3$ *and a positive integer* n *such that* $p \mid n$. *We will use* ${}^UE, {}^TE$, *and* ${}^KE$ *to denote the corresponding* p*-localized Serre spectral sequences.*

3.2. **Nontrivial elements of** ${}^UE$**.** By Proposition 2.2 and equation (2.4), in the range $s \leq 2p+7$, ${}^UE_2^{s,t}$ could be nonzero only when $s = 0, 3, 2p+2$, or $2p+5$, and $t \geq 0$ is even. Therefore, along the line $s + t = 2p+6$ of the E_∞-page, the only places where ${}^UE_\infty^{s,t}$ could possibly be nonzero are ${}^UE_\infty^{0,2p+6}$ and ${}^UE_\infty^{2p+2,4}$. Then the proof of Theorem 1 boils down to proving the following proposition.

Proposition 3.3. *None of the nontrivial elements in* ${}^UE_2^{2p+2,4}$ *could survive to the* E_∞*-page. In other words,* ${}^UE_\infty^{2p+2,4} = 0$.

Proof of Theorem 1 assuming Proposition 3.3. Let us first point out that, by the discussions following Theorem 1, we can feel free to assume $p \geq 3$ and $p \mid n$.

Now, using the Serre spectral sequence ${}^UE$, we get a short exact sequence of $\mathbb{Z}_{(p)}$-modules

$$ 0 \to {}^UE_\infty^{2p+2,4} \to H^{2p+6}(BPU_n)_{(p)} \to {}^UE_\infty^{0,2p+6} \to 0 \tag{3.2} $$

From the isomorphism ${}^UE_2^{s,t} \cong H^s(K(\mathbb{Z},3)) \otimes H^t(BU_n)_{(p)}$, we get

$$ {}^UE_2^{0,2p+6} = H^0(K(\mathbb{Z},3)) \otimes H^{2p+6}(BU_n)_{(p)} \cong H^{2p+6}(BU_n)_{(p)} $$

is the free $\mathbb{Z}_{(p)}$-module generated by monomials in $c_1, c_2, \ldots, c_n$ in dimension $2p+6$. Inspection of degrees shows that ${}^UE_*^{0,2p+6}$ can not receive any nontrivial differentials. Hence ${}^UE_\infty^{0,2p+6} \subset {}^UE_2^{0,2p+6}$ is a free $\mathbb{Z}_{(p)}$-module. Then the short exact sequence (3.2) splits and we get

$$H^{2p+6}(BPU_n)_{(p)} \cong {}^UE_\infty^{2p+2,4} \oplus {}^UE_\infty^{0,2p+6}$$

This implies

$${}_pH^{2p+6}(BPU_n)_{(p)} \subset {}^UE_\infty^{2p+2,4}$$

Now the result follows from Proposition 3.3. □

3.4. **Inspection of ${}^UE_*^{2p+2,4}$.** Note the differentials in ${}^UE$ have the form

$$d_r : {}^UE_r^{s,t} \to {}^UE_r^{s+r,t-r+1}$$

Inspection of degrees shows that ${}^UE_*^{2p+2,4}$ can receive only the d_{2p-1} differential

$$d_{2p-1} : {}^UE_{2p-1}^{3,2p+2} \to {}^UE_{2p-1}^{2p+2,4}$$

and support the d_3 differential

$$d_3 : {}^UE_3^{2p+2,4} \to {}^UE_3^{2p+5,2}$$

By similar arguments, ${}^UE_*^{3,2p+2}$ can receive only the d_3 differential and support the d_{2p-1} differential.

To simplify the notations, we let

$$M^1 = {}^UE_2^{3,2p+2}, M^2 = {}^UE_2^{2p+2,4}, M^3 = {}^UE_2^{2p+5,2}$$

One simple observation is that, since ${}^UE_2$ is concentrated in even rows, all d_2 differentials are trivial. In particular, we also have

$$M^1 = {}^UE_3^{3,2p+2}, M^2 = {}^UE_3^{2p+2,4}, M^3 = {}^UE_3^{2p+5,2}$$

Moreover,

$$(3.3) \qquad {}^UE_{2p-1}^{2p+2,4} = {}^UE_{2p-2}^{2p+2,4} = \cdots = {}^UE_4^{2p+2,4} = Ker(d_3) \subset {}^UE_3^{2p+2,4} = M^2$$

On the other hand,

$$(3.4) \qquad {}^UE_\infty^{2p+2,4} = \cdots = {}^UE_{2p}^{2p+2,4} = {}^UE_{2p-1}^{2p+2,4}/Im(d_{2p-1})$$

Again, to simplify the notations, we let δ^1 denote the composition

$$\delta^1 : M^1 = {}^UE_3^{3,2p+2} \to {}^UE_3^{3,2p+2}/\operatorname{Im} d_3 = {}^UE_{2p-1}^{3,2p+2} \xrightarrow{d_{2p-1}} {}^UE_{2p-1}^{2p+2,4} \subset M^2$$

We let δ^2 denote the map

$$\delta^2 : M^2 = {}^UE_3^{2p+2,4} \xrightarrow{d_3} {}^UE_3^{2p+5,2} = M^3$$

Before we compute δ^1, δ^2, let us write down the explicit $\mathbb{Z}_{(p)}$-module structures of M^1, M^2, and M^3.

Using the isomorphism ${}^UE_2^{s,t} \cong H^s(K(\mathbb{Z},3)) \otimes H^t(BU_n)_{(p)}$, we get

$$M^1 = H^3(K(\mathbb{Z},3)) \otimes H^{2p+2}(BU_n)_{(p)} \cong H^{2p+2}(BU_n)_{(p)}$$

is the free $\mathbb{Z}_{(p)}$-module generated by elements of the form cx_1 where c is a monomial in $c_1, c_2, \ldots, c_n$ in dimension $2p+2$.

We also have

$$M^2 = H^{2p+2}(K(\mathbb{Z},3)) \otimes H^4(BU_n)_{(p)} = \mathbb{Z}_{(p)}\{c_2y_{p,0}, c_1^2y_{p,0}\}/p \cong \mathbb{Z}/p \oplus \mathbb{Z}/p$$

and

$$M^3 = H^{2p+5}(K(\mathbb{Z},3)) \otimes H^2(BU_n)_{(p)} = \mathbb{Z}_{(p)}\{c_1x_1y_{p,0}\}/p \cong \mathbb{Z}/p$$

Now, Proposition 3.3 could be proved using the following two lemmas.

Lemma 3.5. *As a subgroup of M^2, the kernel of $\delta^2 : M^2 \to M^3$ is generated by $c_1^2 y_{p,0}$.*

Lemma 3.6. *The image of $\delta^1 : M^1 \to M^2$ contains the subgroup of M^2 generated by $c_1^2 y_{p,0}$.*

Proof of Proposition 3.3 assuming Lemmas 3.5 and 3.6. We have seen from (3.3) and (3.4) that ${}^U E_{2p-1}^{2p+2,4} = Ker(\delta^2)$ and ${}^U E_\infty^{2p+2,4} = {}^U E_{2p-1}^{2p+2,4}/Im(\delta^1)$. Lemma 3.5 together with 3.6 shows $Ker(\delta^2) \subset Im(\delta^1)$. Therefore, ${}^U E_\infty^{2p+2,4} = 0$. □

3.7. **The proofs of Lemma 3.5 and 3.6.** We first study the kernel of δ^2 and prove Lemma 3.5.

Proof of Lemma 3.5. Recall that

$$M^2 = \mathbb{Z}_{(p)}\{c_2 y_{p,0}, c_1^2 y_{p,0}\}/p \cong \mathbb{Z}/p \oplus \mathbb{Z}/p$$

$$M^3 = \mathbb{Z}_{(p)}\{c_1 x_1 y_{p,0}\}/p \cong \mathbb{Z}/p$$

The map $\delta^2 : M^2 \xrightarrow{d_3} M^3$ is determined by its behavior on the generators.

By inspection of degrees, we have ${}^U d_3(y_{p,0}) = 0$. By Corollary 2.9 combined with the Leibniz rule, we know

$$\delta^2(c_2 y_{p,0}) = d_3(c_2 y_{p,0}) = (n-1) c_1 x_1 y_{p,0} \neq 0 \in M^3$$

$$\delta^2(c_1^2 y_{p,0}) = d_3(c_1^2 y_{p,0}) = 2n c_1 x_1 y_{p,0} = 0 \in M^3$$

Here, recall from Notation 3.1 that we assumed $p \mid n$.

Therefore, the kernel of δ^2 is generated by $c_1^2 y_{p,0}$. □

Now, we analyze the image of $\delta^1 : M^1 \to M^2$ and prove Lemma 3.6. The strategy is to find an explicit preimage of a nontrivial element in $\mathbb{Z}/p\{c_1^2 y_{p,0}\}$. We claim that

$$\delta^1(c_p c_1 x_1) = \binom{n-1}{p-1} c_1^2 y_{p,0}$$

Hence $c_p c_1 x_1 \in M^1$ could serve our purpose.

Proof of Lemma 3.6. We compute $\delta^1(c_p c_1 x_1)$ for the element $c_p c_1 x_1 \in M^1$. Instead of computing this differential directly, we first use the map $\Psi^* : {}^U E \to {}^T E$ of spectral sequences to consider the image of $\delta^1(c_p c_1 x_1)$ in ${}^T E$.

$$\begin{aligned} &\Psi^* \, {}^U d_{2p-1}(c_p c_1 x_1) = {}^T d_{2p-1} \Psi^*(c_p c_1 x_1) \\ &= {}^T d_{2p-1}[(\sum_{n \geq i_1 > i_2 > \cdots > i_p \geq 1} v_{i_1} v_{i_2} \cdots v_{i_p})(v_1 + v_2 + \cdots + v_n) x_1] \end{aligned} \tag{3.5}$$

To simplify the computation, we introduce the new elements $v_i' = v_i - v_n$ for $1 \leq i \leq n$. The advantage is that, by Proposition 2.8(2), the v_i''s are all permanent cocycles. Now, we use the v_i''s and the summation notation σ_i's defined in (2.11) to rewrite the result in (3.5).

$$
\begin{aligned}
&(\sum_{n\geq i_1>i_2>\cdots>i_p\geq 1} v_{i_1}v_{i_2}\cdots v_{i_p})(v_1+v_2+\cdots+v_n)x_1\\
&=[\sum_{n\geq i_1>i_2>\cdots>i_p\geq 1}(v'_{i_1}+v_n)(v'_{i_2}+v_n)\cdots(v'_{i_p}+v_n)](\sum_{k=1}^{n}v'_k+nv_n)x_1\\
&=[\sum_{n\geq i_1>i_2>\cdots>i_p\geq 1}\sum_{j=0}^{p}\sigma_j(v'_{i_1},\cdots,v'_{i_p})v_n^{p-j}](\sum_{k=1}^{n}v'_k+nv_n)x_1\\
&=[\sum_{n\geq i_1>i_2>\cdots>i_p\geq 1}\sum_{j=0}^{p}\sigma_j(v'_{i_1},\cdots,v'_{i_p})v_n^{p-j}][\sum_{k=1}^{n}v'_k]x_1\\
&\quad+n[\sum_{n\geq i_1>i_2>\cdots>i_p\geq 1}\sum_{j=0}^{p}\sigma_j(v'_{i_1},\cdots,v'_{i_p})v_n^{p-j+1}]x_1
\end{aligned}
\tag{3.6}
$$

Now, using Proposition 2.7, we can continue the computations in (3.5) and (3.6)

$$
\begin{aligned}
&\Psi^{*}\ {}^{U}d_{2p-1}(c_pc_1x_1)\\
&={}^{T}d_{2p-1}\{[\sum_{n\geq i_1>i_2>\cdots>i_p\geq 1}\sum_{j=0}^{p}\sigma_j(v'_{i_1},\ldots,v'_{i_p})v_n^{p-j}][\sum_{k=1}^{n}v'_k]x_1\}\\
&\quad+{}^{T}d_{2p-1}\{n[\sum_{n\geq i_1>i_2>\cdots>i_p\geq 1}\sum_{j=0}^{p}\sigma_j(v'_{i_1},\ldots,v'_{i_p})v_n^{p-j+1}]x_1\}\\
&={}^{T}d_{2p-1}\{[\sum_{n\geq i_1>i_2>\cdots>i_p\geq 1}\sigma_1(v'_{i_1},\ldots,v'_{i_p})][\sum_{k=1}^{n}v'_k]v_n^{p-1}x_1\}\\
&\quad+{}^{T}d_{2p-1}\{n[\sum_{n\geq i_1>i_2>\cdots>i_p\geq 1}\sigma_2(v'_{i_1},\ldots,v'_{i_p})v_n^{p-1}]x_1\}\\
&=[\sum_{n\geq i_1>i_2>\cdots>i_p\geq 1}\sigma_1(v'_{i_1},\ldots,v'_{i_p})][\sum_{k=1}^{n}v'_k]y_{p,0}\\
&\quad+n[\sum_{n\geq i_1>i_2>\cdots>i_p\geq 1}\sigma_2(v'_{i_1},\ldots,v'_{i_p})]y_{p,0}
\end{aligned}
\tag{3.7}
$$

Here, we are using the fact that v'_i's are permanent cocycles. Noticing that $y_{p,0}$ is p-torsion and $p \mid n$, we can further simplify the result in (3.7)

$$
\begin{aligned}
&\Psi^{*}\ {}^{U}d_{2p-1}(c_pc_1x_1)\\
&=[\sum_{n\geq i_1>i_2>\cdots>i_p\geq 1}(v'_{i_1}+v'_{i_2}+\cdots+v'_{i_p})][v'_1+v'_2+\cdots+v'_n]y_{p,0}\\
&=[\sum_{n\geq i_1>i_2>\cdots>i_p\geq 1}(v_{i_1}+v_{i_2}+\cdots+v_{i_p})][v_1+v_2+\cdots+v_n]y_{p,0}\\
&=[\binom{n-1}{p-1}\sum_{k=1}^{n}v_k](\sum_{k=1}^{n}v_k)y_{p,0} \qquad (3.8)\\
&=\binom{n-1}{p-1}(\sum_{k=1}^{n}v_k)^2y_{p,0}\\
&=\Psi^*(\binom{n-1}{p-1}c_1^2y_{p,0})
\end{aligned}
$$

Recall that we know the comparison map

$$\Psi^*:{}^{U}E_2^{2p+2,4}\to\ {}^{T}E_2^{2p+2,4}$$

is injective (2.10). We also know ${}^{U}E_{2p-1}^{2p+2,4}$ is a subgroup of ${}^{U}E_2^{2p+2,4}$ (3.3). Similar argument shows ${}^{T}E_{2p-1}^{2p+2,4}$ is a subgroup of ${}^{T}E_2^{2p+2,4}$. Hence the induced map

$$\Psi^*:{}^{U}E_{2p-1}^{2p+2,4}\to\ {}^{T}E_{2p-1}^{2p+2,4}$$

is also injective. Then (3.8) shows

$$\delta^1(c_pc_1x_1)=\ {}^{U}d_{2p-1}(c_pc_1x_1)=\binom{n-1}{p-1}c_1^2y_{p,0}$$

Note $\binom{n-1}{p-1}$ is coprime to p, this shows the image of $\delta^1: M^1\to M^2$ contains the subgroup of M^2 generated by $c_1^2y_{p,0}$.

□

3.8. **The computation of ${}_pH^{2p+5}(BPU_n)$.** At the conclusion of this paper, we would like to discuss the current state of the open problem regarding the computation of ${}_pH^{2p+5}(BPU_n)$. The primary challenge in determining this value is the calculation of ${}^{U}E_{2p-2}^{3,2p+2}$, which, due to degree considerations, is equivalent to ${}^{U}E_4^{3,2p+2}$. Thus, the problem is reduced to determining the image of ${}^{U}d_3^{0,2p+4}:{}^{U}E_3^{0,2p+4}\to{}^{U}E_3^{3,2p+2}$. Here, we have

$$
\begin{aligned}
{}^{U}E_3^{0,2p+4}&={}^{U}E_2^{0,2p+4}=H^0(K(\mathbb{Z},3))\otimes H^{2p+4}(BU_n)_{(p)}\cong H^{2p+4}(BU_n)_{(p)},\\
{}^{U}E_3^{3,2p+2}&={}^{U}E_2^{3,2p+2}=H^3(K(\mathbb{Z},3))\otimes H^{2p+2}(BU_n)_{(p)}\cong H^{2p+2}(BU_n)_{(p)}.
\end{aligned}
$$

It is worth noting that the method presented in [10] cannot be used directly to compute the image of ${}^{U}d_3^{0,2p+4}$. We will now explain the reasons for this.

Recall that the $\mathbb{Z}_{(p)}$-module ${}^{U}E_3^{3,2p}$ is generated freely by elements of the form cx_1, where c is an element of the set S' defined as $S':=\{c_1^{i_1}c_2^{i_2}\cdots c_p^{i_p}\mid i_k\geq 0,\ \sum_k ki_k=p\}$. In [10], the authors computed $\operatorname{Im}\delta^0:=\operatorname{Im}{}^{U}d_3^{0,2p+2}:{}^{U}E_3^{0,2p+2}\to{}^{U}E_3^{3,2p}$ by introducing a total ordering on S', which also induces total orderings on $S'x_1$ and $S:=S'-\{c_p\}$.

Let L be the $\mathbb{Z}_{(p)}$-submodule of $H^{2p}(BU_n)_{(p)}$ spanned by S. Consider the $\mathbb{Z}_{(p)}$-linear map $\tau : L \to {}^U E_3^{0,2p+2} = H^{2p+2}(BU_n)_{(p)}$ defined by

$$\tau(c_1^{i_1} c_2^{i_2} \cdots c_k^{i_k}) = c_1^{i_1} c_2^{i_2} \cdots c_{k-1}^{i_{k-1}} c_k^{i_k - 1} c_{k+1}.$$

For any element $c := c_1^{i_1} \cdots c_k^{i_k} \in S$, we have

$$\delta^0 \tau(c) \equiv (n-k) c x_1 + \text{higher order terms} \mod p.$$

Therefore, the associated coefficient matrix takes the form

$$A \equiv \begin{pmatrix} \lambda_1 & * & \cdots & * \\ 0 & \lambda_2 & * & \vdots \\ 0 & 0 & \ddots & * \\ 0 & \cdots & 0 & \lambda_N \\ 0 & 0 & \cdots & 0 \end{pmatrix} \pmod p,$$

where the λ_i's are of the form $n-k$ for $k < p$, and hence invertible in $\mathbb{Z}_{(p)}$. Based on this, we can determine $\operatorname{Im} \delta^0 = \operatorname{Im} {}^U d_3^{0,2p+2}$.

Now, we will attempt to use similar methods to calculate $\operatorname{Im} {}^U d_3^{0,2p+4}$. In this case, the set S' becomes

$$S' = \{c_1^{i_1} c_2^{i_2} \cdots c_{p+1}^{i_{p+1}} \mid i_k \geq 0, \ \sum_k k i_k = p+1\}.$$

Then, in the corresponding coefficient matrix A, the λ_i's are of the form $n-k$ for $k < p+1$, which is no longer invertible in $\mathbb{Z}_{(p)}$ if $k = p$. Therefore, similar computation strategies can not be directly applied to compute ${}_pH^{2p+5}(BPU_n)$.

References

[1] Benjamin Antieau and Ben Williams. The period-index problem for twisted topological K-theory. *Geometry & Topology*, 18(2):1115–1148, 2014.

[2] Benjamin Antieau and Ben Williams. The topological period–index problem over 6-complexes. *Journal of Topology*, 7(3):617–640, 2014.

[3] Henri Cartan and Jean-Pierre Serre. Séminaire Henri Cartan vol. 7. pages 283–288, 1954-1955.

[4] Clay Cordova, Daniel Freed, Ho Tat Lam, and Nathan Seiberg. Anomalies in the space of coupling constants and their dynamical applications II. *SciPost Physics Proceedings*, 8(1), 2020.

[5] Haibao Duan. The cohomology and K-theory of the projective unitary groups $PU(n)$. *arXiv preprint arXiv:1710.09222*, 2020.

[6] Iñaki García-Etxebarria and Miguel Montero. Dai-Freed anomalies in particle physics. *Journal of High Energy Physics*, 2019(8):3, 2019.

[7] Xing Gu. The topological period–index problem over 8-complexes, I. *Journal of Topology*, 12(4):1368–1395, 2019.

[8] Xing Gu. The topological period-index problem over 8-complexes, II. *Proceedings of the American Mathematical Society*, 148:4541–4545, 2020.

[9] Xing Gu. On the cohomology of the classifying spaces of projective unitary groups. *Journal of Topology and Analysis*, 13(02):535–573, 2021.

[10] Xing Gu, Yu Zhang, Zhilei Zhang, and Linan Zhong. The p-primary subgroups of the cohomology of BPU_n in dimensions less than $2p+5$. *Proc. Amer. Math. Soc.*, 150(9):4099–4111, 2022.

[11] Masaki Kameko and Nobuaki Yagita. The Brown-Peterson cohomology of the classifying spaces of the projective unitary groups $PU(p)$ and exceptional Lie groups. *Transactions of the American Mathematical Society*, 360(5):2265–2284, 2008.

[12] Akira Kono and Mamoru Mimura. On the cohomology of the classifying spaces of $PSU(4n+2)$ and $PO(4n+2)$. *Publications of the Research Institute for Mathematical Sciences*, 10(3):691–720, 1975.
[13] Akira Kono and Nobuaki Yagita. Brown-Peterson and ordinary cohomology theories of classifying spaces for compact Lie groups. *Trans. Amer. Math. Soc.*, 339(2):781–798, 1993.
[14] Hirotaka Tamanoi. Q-subalgebras, Milnor basis, and cohomology of Eilenberg-MacLane spaces. *Journal of Pure and Applied Algebra*, 137(2):153–198, 1999.
[15] Hiroshi Toda et al. Cohomology of classifying spaces. In *Homotopy theory and related topics*, pages 75–108. Mathematical Society of Japan, 1987.
[16] Aleš Vavpetič and Antonio Viruel. On the mod p cohomology of $BPU(p)$. *Transactions of the American Mathematical Society*, pages 4517–4532, 2005.

Department of Mathematics, Nankai University, No.94 Weijin Road, Tianjin 300071, P. R. China
Email address: zhang.4841@buckeyemail.osu.edu

Department of Mathematics, Nankai University, No.94 Weijin Road, Tianjin 300071, P. R. China
Email address: 15829207515@163.com

Department of Mathematics, Yanbian University, No. 997 Gongyuan Road, 133000, Yanji, Jilin Province, People's Republic of China
Email address: lnzhong@ybu.edu.cn