THE p-PRIMARY SUBGROUPS OF THE COHOMOLOGY OF BPU_n IN DIMENSIONS LESS THAN $2p+5$

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Abstract. Let PU_n denote the projective unitary group of rank n and BPU_n be its classifying space. For an odd prime p, we extend previous results to a complete description of $H^s(BPU_n;\mathbb{Z})_{(p)}$ for $s < 2p+5$ by showing that the p-primary subgroups of $H^s(BPU_n;\mathbb{Z})$ are trivial for $s = 2p+3$ and $s = 2p+4$.

1. Introduction

Let U_n denote the group of $n \times n$ unitary matrices. The unit circle S^1 can be viewed as the normal subgroup of scalar matrices of U_n. We let PU_n denote the quotient group of U_n by S^1, and BPU_n be the classifying space of PU_n. In this paper we consider $H^*(BPU_n;\mathbb{Z})$, th7e ordinary cohomology of BPU_n with coefficients in $\mathbb{Z}$.

A review of the literature. The ordinary and generalized cohomology of BPU_n for special n has been the subject of various works such as Kono-Mimura [15], Kameko-Yagita [14], Kono-Yagita [16], Toda [19], and Vavpetič-Viruel [21]. Vezzosi [22] and Vistoli [23] studied the Chow ring of the classifying space (in the sense of Totaro [20]) of $BPGL_3(\mathbb{C})$ and $BPGL_p(\mathbb{C})$ for p an odd prime, respectively. Much of their results applies to the ordinary cohomology of BPU_p.

None of the works above dealt with $H^*(BPU_n;\mathbb{Z})$ for n not a prime number. The first named author considered $H^*(BPU_n;\mathbb{Z})$, as well as the Chow ring of $BPGL_n(\mathbb{C})$ for an arbitrary n in [10], [12] and [13]. In particular, in [12], the first named author determined the ring structure of $H^*(BPU_n;\mathbb{Z})$ in dimensions less than or equal to 10.

Other related works include Duan [6], in which the integral cohomology of PU_n is fully determined, and Crowley-Gu [5], in which the image of the canonical map $H^*(BPU_n;\mathbb{Z}) \to H^*(BU_n;\mathbb{Z})$ is studied.

The cohomology of BPU_n plays significant roles in the study of the topological period-index problem ([1], [2], [9] and [11]), and in the study of anormalies in partical physics ([4], [8]).

Notations. Throughout the rest of this paper, $H^*(-)$ denotes $H^*(-;\mathbb{Z})$. For an abelian group A and a prime number p, let $A_{(p)}$ be the localization of A at p, and let ${}_pA$ denotes the p-primary subgroup of A, i.e., the subgroup of A of all torsion elements with torsion order a power of p. In particular, we have a canonical isomorphism ${}_pH^*(-) \cong {}_p[H^*(-)_{(p)}]$, and we will not distinguish the two throughout this paper. Tensor products of $\mathbb{Z}_{(p)}$-modules are always taken over $\mathbb{Z}_{(p)}$.

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The main theorem and some remarks. We review a basic fact on the cohomology of BPU_n. Consider the short exact sequence of Lie groups

$$1 \to \mathbb{Z}/n \to SU_n \to PSU_n \simeq PU_n \to 1,$$

which induces a fiber sequence of their classifying spaces

$$B(\mathbb{Z}/n) \to BSU_n \to BPU_n \tag{1.1}$$

When $p \nmid n$, the space $B(\mathbb{Z}/n)$ is p-locally contractible and we have

$$H^*(BPU_n; \mathbb{Z}_{(p)}) \cong H^*(BSU_n; \mathbb{Z}_{(p)}) \tag{1.2}$$

Since $\mathbb{Z}_{(p)}$ is a flat $\mathbb{Z}$-module, and in particular, $H^*(-; \mathbb{Z}_{(p)}) \cong H^*(-)_{(p)}$, we have an isomorphism of $\mathbb{Z}_{(p)}$-algebras

$$H^*(BPU_n)_{(p)} \cong H^*(BSU_n)_{(p)} = \mathbb{Z}_{(p)}[c_2, c_3, \ldots, c_n], \tag{1.3}$$

which shows $H^*(BPU_n)_{(p)}$ is torsion-free for $p \nmid n$. In other words, we have the following

Proposition 1.1. *Suppose $x \in H^*(BPU_n)$ is a torsion class. Then there exists some $i \geq 0$ such that $n^i x = 0$.*

Therefore, to determine the graded abelian group structure of $H^s(BPU_n)$, it suffices to consider the p-primary subgroup ${}_pH^s(BPU_n)$ for $p \mid n$.

Remark 1.2. In the case of Chow rings, Vezzosi [22] proved the stronger result that all torsion classes in the Chow ring of $BPGL_n(\mathbb{C})$ are n-torsion.

To state the main theorem, recall that, as shown in [12], the integral cohomology group $H^3(BPU_n)$ is generated by a class denoted by x_1. In addition, P^i will denote the ith Steenrod reduced power operation, and

$$\delta : H^*(-, \mathbb{Z}/p) \to H^{*+1}(-)$$

will denote the connecting homomorphism. Finally, a bar over an integral cohomology class will denote the mod p reduction of this class. For instance, $\bar{x}_1$ denotes the mod p reduction of x_1, which is in $H^3(BPU_n; \mathbb{Z}/p)$.

Theorem 1. *Let $p > 2$ be a prime number, and $n = p^r m$ for a positive integer m co-prime to p. Then the p-primary subgroup of $H^s(BPU_n)$ in dimensions less than $2p+5$ is as follows:*

1. *For $r > 0$, we have*

$$ {}_pH^s(BPU_n) \cong \begin{cases} \mathbb{Z}/p^r, & s = 3, \\ \mathbb{Z}/p, & s = 2p+2, \\ 0, & s < 2p+5, \ s \neq 3, 2p+2. \end{cases} $$

   The group ${}_pH^{2p+2}(BPU_n)$ is generated by $\delta\,\mathrm{P}^1(\bar{x}_1)$.

2. *For $r = 0$, we simply have ${}_pH^s(BPU_n) = 0$ for all $s \geq 0$.*

Remark 1.3. Note ${}_pH^s(BPU_n) \cong {}_pH^s(BPU_n)_{(p)}$. By the discussion preceding Remark 1.2, Theorem 1 completely determines $H^s(BPU_n; \mathbb{Z}_{(p)})$ for $0 \leq s < 2p+5$.

For $s \leq 3$, the groups ${}_pH^s(BPU_n)$ are well known and are part of Theorem 1.1 of [12]. For $3 < s < 2p+2$, they are given in Theorem 1.2 of [12]. Therefore, what remains to show is

$$ {}_pH^{2p+2}(BPU_n) \cong \mathbb{Z}/p, \ {}_pH^{2p+3}(BPU_n) = {}_pH^{2p+4}(BPU_n) = 0. \tag{1.4}$$

Remark 1.4. For $p = 2$, it was shown by the first named author [12] that the 2-torsion subgroup of $H^s(BPU_n)$ in dimension $s = 2p+3 = 7$ is $\mathbb{Z}/2$ if $n \equiv 2 \bmod 4$, and is 0 otherwise. In particular, Theorem 1 does not generalize to the case $p = 2$.

Remark 1.5. For $p = 3$, (1.4) follows immediately from the computation in [12] of $H^s(BPU_n)$ in dimensions $8, 9$ and 10.

Organization of the paper. In Section 2, we discuss some preliminary results of the Serre spectral sequence ${}^UE$ associated to the fiber sequence $U: \ BU_n \to BPU_n \to K(\mathbb{Z}, 3)$. This will be our main tool for computing the p-primary subgroup ${}_pH^s(BPU_n)$. We will also show that (1.4) can be deduced from Theorem 1.2 of [12] and Proposition 2.10, which says that certain chain complex $\mathcal{M}$ constructed from the differentials in ${}^UE$ is exact.

In Section 3, we prove Proposition 2.10. The proof is based on the explicit computation of some relevant differentials in ${}^UE$. This section finishes our proof of Theorem 1.

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2. The spectral sequences

The Serre spectral sequence ${}^UE$. We follow the strategy employed in [12] to compute the cohomology of BPU_n. The short exact sequence of Lie groups

$$1 \to S^1 \to U_n \to PU_n \to 1$$

induces a fiber sequence of their classifying spaces

$$BS^1 \to BU_n \to BPU_n.$$

Notice that BS^1 is of the homotopy type of the Eilenberg-Mac Lane space $K(\mathbb{Z}, 2)$ and indeed we obtain another fiber sequence

$$U: \ BU_n \to BPU_n \xrightarrow{\chi} K(\mathbb{Z}, 3). \tag{2.1}$$

Remark 2.1. In general, it is not always possible to obtain a fiber sequence of the form $F \to E \to B$ from a fiber sequence $\Omega B \to F \to E$. See Ganea [7] for more.

We will use the Serre spectral sequence associated to the last fiber sequence to compute the cohomology of BPU_n. For notational convenience, we denote this spectral sequence by ${}^UE$. The E_2 page of ${}^UE$ has the form

$${}^UE_2^{s,t} = H^s(K(\mathbb{Z}, 3); H^t(BU_n)) \Longrightarrow H^{s+t}(BPU_n).$$

In principle, Cartan and Serre [3] determined the cohomology of $K(A,n)$ for all finitely generated abelian groups A. Also see Tamanoi [18] for a nice treatment.

We summarize the p-local cohomology of $K(\mathbb{Z},3)$ in low dimensions as follows.

Proposition 2.2. *Let $p>2$ be a prime. In degrees up to $2p+5$, we have*

$$H^s(K(\mathbb{Z},3))_{(p)} = \begin{cases} \mathbb{Z}_{(p)}, & s=0,\ 3,\\ \mathbb{Z}/p, & s=2p+2,\ 2p+5,\\ 0, & s<2p+5, s\neq 0,\ 3,\ 2p+2. \end{cases} \tag{2.2}$$

where $x_1,\ y_{p,0},\ x_1y_{p,0}$ are generators on degree $3, 2p+2, 2p+5$ respectively. In addition, we have $y_{p,0}=\delta\,\mathrm{P}^1(\bar{x}_1)$.

Here we use the same notations for the generators as in [12]. Sometimes we abuse notations and let $x_1, y_{p,0}$ denote $\chi^*(x_1)$, $\chi^*(y_{p,0})$, where $\chi: BPU_n\to K(\mathbb{Z},3)$ is defined in (2.1). For instance, we have

Proposition 2.3 (Theorem 1.2, [12])**.** *Let p be a prime. In $H^{2p+2}(BPU_n)$, we have $y_{p,0}\neq 0$ of order p when $p\mid n$, and $y_{p,0}=0$ otherwise. Furthermore, the p-torsion subgroup of $H^k(BPU_n)$ is 0 for $3<k<2p+2$.*

Also recall

$$H^*(BU_n)=\mathbb{Z}[c_1,c_2,\dots,c_n],\ |c_i|=2i. \tag{2.3}$$

In particular, $H^*(BU_n)$ is torsion-free. We have

$${}^UE_2^{s,t}\cong H^s(K(\mathbb{Z},3))\otimes H^t(BU_n).$$

The auxiliary fiber sequences and spectral sequences. To determine some of the differentials in ${}^UE$, we consider two more fiber sequences.

Let T^n be the maximal torus of U^n with the inclusion denoted by

$$\psi: T^n\to U_n.$$

Passing to quotients over S^1, we have another inclusion of maximal torus

$$\psi': PT^n\to PU_n.$$

The quotient map $T^n\to PT^n$ fits in an exact sequnce of Lie groups

$$1\to S^1\to T^n\to PT^n\to 1,$$

which induces a fiber sequence

$$T:\ BT^n\to BPT^n\to K(\mathbb{Z},3).$$

Notice that we have

$$H^*(BT^n)=\mathbb{Z}[v_1,v_2,\dots,v_n],\ |v_i|=2. \tag{2.4}$$

The next fiber sequence is simply the path fibration for the space $K(\mathbb{Z},3)$

$$K:\ K(\mathbb{Z},2)\to *\to K(\mathbb{Z},3)$$

where $*$ denotes a contractible space. We denote their associated Serre spectral sequences as ${}^TE$ and ${}^KE$ respectively.

We denote the corresponding differentials of ${}^UE$, ${}^TE$, and ${}^KE$ by ${}^Ud_*^{*,*}$, ${}^Td_*^{*,*}$, and ${}^Kd_*^{*,*}$, respectively, if there are risks of ambiguity. Otherwise, we simply denote the differentials by $d_*^{*,*}$.

These fiber sequences fit into the following homotopy commutative diagram:

$$(2.5)\qquad \begin{array}{ccccccc} K: & & K(\mathbb{Z},2) & \longrightarrow & * & \longrightarrow & K(\mathbb{Z},3) \\ \downarrow{\scriptstyle\Phi} & & \downarrow{\scriptstyle B\varphi} & & \downarrow & & \downarrow{\scriptstyle =} \\ T: & & BT^n & \longrightarrow & BPT^n & \longrightarrow & K(\mathbb{Z},3) \\ \downarrow{\scriptstyle\Psi} & & \downarrow{\scriptstyle B\psi} & & \downarrow{\scriptstyle B\psi'} & & \downarrow{\scriptstyle =} \\ U: & & BU_n & \longrightarrow & BPU_n & \longrightarrow & K(\mathbb{Z},3) \end{array}$$

Here, the map $B\varphi : K(\mathbb{Z},2) \simeq BS^1 \to BT^n$ is the de-looping of the diagonal map $S^1 \to T^n$. The induced homomorphism between cohomology rings is as follows:

$$B\varphi^* : H^*(BT^n) = \mathbb{Z}[v_1, v_2, \cdots, v_n] \to H^*(BS^1) = \mathbb{Z}[v],\ \ v_i \mapsto v.$$

The map $B\psi : BT^n \to BU_n$ induces the injective ring homomorphism

$$B\psi^* : H^*(BU_n) = \mathbb{Z}[c_1, \cdots, c_n] \to H^*(BT^n) = \mathbb{Z}[v_1, \cdots, v_n],$$
$$c_i \mapsto \sigma_i(v_1, \cdots, v_n),$$

where $\sigma_j(t_1, t_2, \cdots, t_n)$ be the jth elementary symmetric polynomial in variables $t_1, t_2, \cdots, t_n$:

$$(2.6)\qquad \begin{aligned} &\sigma_0(t_1, t_2, \cdots, t_n) = 1,\\ &\sigma_1(t_1, t_2, \cdots, t_n) = t_1 + t_2 + \cdots + t_n,\\ &\sigma_2(t_1, t_2, \cdots, t_n) = \sum_{i<j} t_i t_j,\\ &\vdots\\ &\sigma_p(t_1, t_2, \cdots, t_n) = t_1 t_2 \cdots t_n. \end{aligned}$$

We will use the associated maps of spectral sequences to compute the differentials in ${}^UE$. This is possible because we have a good understanding of the corresonding differentials in ${}^TE$ and ${}^KE$. In particular, we have the following results.

Proposition 2.4 ([12], Corollary 2.16)**.** *The higher differentials of* ${}^KE_*^{*,*}$ *satisfy*

$$\begin{aligned} &d_3(v) = x_1,\\ &d_{2p-1}(x_1 v^{lp^e-1}) = v^{lp^e-1-(p-1)} y_{p,0}, \quad e > 0,\ \gcd(l,p) = 1,\\ &d_r(x_1) = d_r(y_{p,0}) = 0, \quad \textit{for all } r, \end{aligned}$$

and the Leibniz rule.

Remark 2.5. Proposition 2.4 is a special case of Corollary 2.16, [12]. Here, we take the opportunity to correct a typo in the original Corollary 2.16, [12], where the condition $k \geq e$ should be replaced by $e > k$.

Proposition 2.6 ([12], Proposition 3.2)**.** *The differential* ${}^Td_r^{*,*}$, *is partially determined as follows:*

$$(2.7)\qquad {}^Td_r^{*,2t}(v_i^t \xi) = (B\pi_i)^*({}^Kd_r^{*,2t}(v^t \xi)),$$

where $\xi \in {}^TE_r^{*,0}$, *a quotient group of* $H^*(K(\mathbb{Z},3))$, *and* $\pi_i : T^n \to S^1$ *is the projection of the* i*th diagonal entry. In plain words,* ${}^Td_r^{*,2t}(v_i^t\xi)$ *is simply* ${}^Kd_r^{*,2t}(v^t\xi)$ *with* v *replaced by* v_i.

Remark 2.7. Here we correct another typo in the original Proposition 3.2 in [12], in which " $\xi \in {}^TE_r^{0,*}$ " should be replaced by " $\xi \in {}^TE_r^{*,0}$ ".

Proposition 2.8 ([12], Proposition 3.3)**.** (1) *The differential* ${}^Td_3^{0,t}$ *is given by the "formal divergence"*

$$\nabla = \sum_{i=1}^{n}(\partial/\partial v_i) : H^t(BT^n; R) \to H^{t-2}(BT^n; R),$$

in such a way that ${}^Td_3^{0,*} = \nabla(-) \cdot x_1$*. For any ground ring* $R = \mathbb{Z}$ *or* $\mathbb{Z}/m$ *for any integer* m*.*

(2) *The spectral sequence degenerates at* ${}^TE_4^{0,*}$*. Indeed, we have* ${}^TE_\infty^{0,*} = {}^TE_4{}^{0,*} = \operatorname{Ker} {}^Td_3^{0,*} = \mathbb{Z}[v_1 - v_n, \cdots, v_{n-1} - v_n]$*.*

Corollary 2.9 ([12], Corollary 3.4)**.**

$${}^Ud_3^{0,*}(c_k) = \nabla(c_k)x_1 = (n-k+1)c_{k-1}x_1.$$

Computations in the spectral sequence ${}^UE$**.** In order to study

$${}_pH^*(BPU_n) \cong {}_p[H^*(BPU_n)_{(p)}],$$

it suffices to look at the p-localized spectral sequence, where the E_2 page becomes

$$({}^UE_2^{s,t})_{(p)} = H^s(K(\mathbb{Z},3))_{(p)} \otimes H^t(BU_n) = H^s(K(\mathbb{Z},3)) \otimes H^t(BU_n)_{(p)}. \tag{2.8}$$

By abuse of notation, for the rest of this paper, we let ${}^UE$, ${}^TE$ and ${}^KE$ denote the corresponding p-localized Serre spectral sequences.

By Proposition 2.2 and (2.3), in the range $s \leq 2p+5$, the only cases in which ${}^UE_2^{s,t}$ could be nonzero are when $s = 0, 3, 2p+2, 2p+5$ and t is even.

To simplify the notations, we let

$$M^0 = {}^UE_2^{0,2p+2},\ M^1 = {}^UE_2^{3,2p},\ M^2 = {}^UE_2^{2p+2,2},\ M^3 = {}^UE_2^{2p+5,0}.$$

Inspection of degrees shows that ${}^UE_*^{3,2p}$ can receive only the d_3 differential and support the d_{2p-1} differential. Similarly, ${}^UE_*^{2p+2,2}$ can receive only the d_{2p-1} differential and support the d_3 differential. In addition, all d_2's are trivial and therefore we have ${}^UE_2^{*,*} = {}^UE_3^{*,*}$.

We let δ^0 be the map

$$\delta^0 : M^0 = {}^UE_3^{0,2p+2} \xrightarrow{d_3} {}^UE_3^{3,2p} = M^1.$$

We let δ^1 be the composition

$$\delta^1 : M^1 = {}^UE_3^{3,2p} \to {}^UE_3^{3,2p}/\operatorname{Im} d_3 = {}^UE_{2p-1}^{3,2p} \xrightarrow{d_{2p-1}} {}^UE_{2p-1}^{2p+2,2} = \operatorname{Ker} d_3 \subset M^2.$$

We let δ^2 be the map

$$\delta^2 : M^2 = {}^UE_3^{2p+2,2} \xrightarrow{d_3} {}^UE_3^{2p+5,0} = M^3.$$

One immediately sees that

$$M^0 \xrightarrow{\delta^0} M^1 \xrightarrow{\delta^1} M^2 \xrightarrow{\delta^2} M^3$$

is a chain complex of $\mathbb{Z}_{(p)}$-modules, which we denote by $\mathcal{M}$. We will show later that Theorem 1 is a consequence of the following

Proposition 2.10. *Let* $p \geq 3$ *be a prime number such that* $p \mid n$*. The chain complex* $\mathcal{M}$ *defined above is exact.*

Proof of Theorem 1 assuming Proposition 2.10. Let $n = p^r m$. For $r = 0$, the theorem follows from Proposition 1.1. In the rest of the proof we assume $r > 0$. First, we prove

$$_pH^{2p+2}(BPU_n) \cong \mathbb{Z}/p.$$

By Proposition 2.3, $y_{p,0} \in^U E_2^{2p+2,0}$ survives to a nonzero element in $H^{2p+2}(BPU_n)$ of order p. Therefore, we have

$${}^UE_\infty^{2p+2,0} = {}^UE_2^{2p+2,0} \cong \mathbb{Z}/p.$$

Since the only nontrivial entries in ${}^UE_2^{*,*}$ of total degree $2p+2$ are ${}^UE_2^{2p+2,0}$ and ${}^UE_2^{0,2p+2}$, we have a short exact sequence of $\mathbb{Z}_{(p)}$-modules

$$0 \to {}^UE_\infty^{2p+2,0} \to H^{2p+2}(BPU_n)_{(p)} \to {}^UE_\infty^{0,2p+2} \to 0.$$

Since ${}^UE_\infty^{0,2p+2} \subset {}^UE_2^{0,2p+2}$ is a free $\mathbb{Z}_{(p)}$-module, the above short exact sequence splits and we have

$$H^{2p+2}(BPU_n)_{(p)} \cong {}^UE_\infty^{2p+2,0} \oplus {}^UE_\infty^{0,2p+2},$$

from which we deduce

$$_pH^{2p+2}(BPU_n) \cong {}^UE_\infty^{2p+2,0} \cong \mathbb{Z}/p.$$

Since the row $E_\infty^{*,0}$ is the image of χ^*, the above implies

$$_pH^{2p+2}(BPU_n) = \chi^*(H^{2p+2}(K(\mathbb{Z}, 3))). \tag{2.9}$$

From (2.9) and Proposition 2.2, it follows that $_pH^{2p+2}(BPU_n)$ is generated by $\delta\,\mathrm{P}^1(\bar{x}_1)$.

Next, we prove

$$_pH^{2p+3}(BPU_n) = H^{2p+3}(BPU_n)_{(p)} = 0.$$

The exactness of $\mathcal{M}$ at M^1 implies ${}^UE_\infty^{3,2p} = 0$. On the other hand, ${}^UE_2^{3,2p}$ is the only nontrivial entry in ${}^UE_2^{*,*}$ of total degree $2p+3$. Hence, we have

$$_pH^{2p+3}(BPU_n) \subset H^{2p+3}(BPU_n)_{(p)} = {}^UE_\infty^{3,2p} = 0.$$

Finally, we prove

$$_pH^{2p+4}(BPU_n) = 0.$$

The exactness of $\mathcal{M}$ at M^2 implies ${}^UE_\infty^{2p+2,2} = 0$. Since ${}^UE_2^{0,2p+4}$ and ${}^UE_2^{2p+2,2}$ are the only nontrivial entries in ${}^UE_2^{*,*}$ of total degree $2p+4$, we have

$$H^{2p+4}(BPU_n)_{(p)} \cong {}^UE_\infty^{0,2p+4},$$

which is torsion-free. In particular, we have $_pH^{2p+4}(BPU_n) = 0$.

□

The proof of Proposition 2.10 occupies Section 3.

3. The proof of Proposition 2.10

From (2.8), we can write out the $\mathbb{Z}_{(p)}$-modules M^0, M^1, M^2, M^3 more explicitly:

$$M^0 = H^0(K(\mathbb{Z},3)) \otimes H^{2p+2}(BU_n)_{(p)} \cong H^{2p+2}(BU_n)_{(p)}$$

is the free $\mathbb{Z}_{(p)}$-module generated by monomials in $c_1, \cdots, c_{p+1}$ in dimension $2p+2$, and

$$M^1 = H^3(K(\mathbb{Z},3)) \otimes H^{2p}(BU_n)_{(p)} \cong H^{2p}(BU_n)_{(p)}$$

is the free $\mathbb{Z}_{(p)}$-module generated by elements of the form cx_1 where c is a monomial in $c_1, \cdots, c_p$ in dimension $2p$. Furthermore, we have

$$M^2 = H^{2p+2}(K(\mathbb{Z},3)) \otimes H^2(BU_n)_{(p)} = \mathbb{Z}_{(p)}\{c_1 y_{p,0}\}/p \cong \mathbb{Z}/p$$

and

$$M^3 = H^{2p+5}(K(\mathbb{Z},3)) \otimes H^0(BU_n)_{(p)} = \mathbb{Z}_{(p)}\{x_1 y_{p,0}\}/p \cong \mathbb{Z}/p.$$

The exactness of $\mathcal{M}$ at M^2.

Lemma 3.1. *In the spectral sequence ${}^TE$, we have*

$$\begin{cases} v_n^k x_1 \in \operatorname{Im} {}^Td_3, \ 0 \le k \le p-2 \ \text{or} \ k = p, \\ {}^Td_{2p-1}^{3,*}(v_n^{p-1}x_1) = y_{p,0}. \end{cases} \tag{3.1}$$

Proof. When $p \nmid k+1$, the first formula in Proposition 2.4 together with Proposition 2.6 imply that

$$v_n^k x_1 = \frac{1}{k+1} {}^Td_3(v_n^{k+1})$$

is in the image of ${}^Td_3$. This completes the proof for the case $0 \le k \le p-2$ or $k = p$.

The remaining case is proved by applying the second formula in Proposition 2.4, taking $e = l = 1$, and then Proposition 2.6. □

Lemma 3.2. *The map $\delta^1 : M^1 \to M^2 \cong \mathbb{Z}/p$ is surjective.*

Proof. Recall the morphism of fiber sequences Ψ introduced in (2.5), and the induced morphism $\Psi^* : {}^UE \to {}^TE$ of spectral sequences.

For $1 \le i \le n$, let $v_i' = v_i - v_n$. It follows from (2) of Proposition 2.8 that the v_i''s are permanent cycles. To determine the value of δ^1 at $c_p x_1 \in M^1$, we have

$$\begin{aligned} &\Psi^*\delta^1(c_p x_1) \\ &= \Psi^* \, {}^Ud_{2p-1}^{3,2p}(c_p x_1) = {}^Td_{2p-1}^{3,2p}\Psi^*(c_p x_1) \\ &= {}^Td_{2p-1}^{3,2p}(\sum_{n\ge i_1>i_2>\ldots>i_p\ge 1.} v_{i_1}v_{i_2}\ldots v_{i_p}x_1) \\ &= {}^Td_{2p-1}^{3,2p}(\sum_{n\ge i_1>i_2>\ldots>i_p\ge 1} (v_{i_1}'+v_n)(v_{i_2}'+v_n)\ldots(v_{i_p}'+v_n)x_1) \\ &= {}^Td_{2p-1}^{3,2p}(\sum_{n\ge i_1>i_2>\ldots>i_p\ge 1}\sum_{j=0}^{p}\sigma_j(v_{i_1}',\cdots,v_{i_p}')v_n^{p-j}x_1)). \end{aligned} \tag{3.2}$$

where $\Psi^* : {}^UE \to {}^TE$ is the morphism of spectral sequences induced by the inclusions of maximal tori $T^n \to U_n$ and $PT^n \to PU_n$, as introduced in (2.5), and σ_i the elementary symmetric polynomials in p variables, as in (2.6).

By Lemma 3.1, we simplify (3.2) and obtain

$$\Psi^*\delta^1(c_px_1) = {}^Td_{2p-1}(\sum_{n\geq i_1>i_2>\ldots>i_p\geq 1}\sigma_1(v'_{i_1},\cdots,v'_{i_p})v_n^{p-1}x_1). \tag{3.3}$$

To proceed, we evaluate the expression

$$\sum_{n\geq i_1>i_2>\ldots>i_p\geq 1}\sigma_1(t_{i_1},\cdots,t_{i_p})$$

for variables $t_i,\ 1\leq i\leq n$. Since it is multi-linear and symmetric in the variables $t_1,\cdots,t_n$, we have

$$\sum_{n\geq i_1>i_2>\ldots>i_p\geq 1}\sigma_1(t_{i_1},\cdots,t_{i_p}) = \lambda\sum_{i=1}^n t_i$$

for some $\lambda\in\mathbb{Z}$. Taking the substitution $t_1=\cdots t_n=1$ and comparing both sides of the above, we obtain

$$\lambda = \frac{p}{n}\binom{n}{p} = \binom{n-1}{p-1}\not\equiv 0 \pmod p$$

and

$$\sum_{n\geq i_1>i_2>\ldots>i_p\geq 1}\sigma_1(t_{i_1},\cdots,t_{i_p}) = \binom{n-1}{p-1}\sum_{i=1}^n t_i. \tag{3.4}$$

Consider the following commutative diagram:

$$\begin{array}{ccc}
M^1 = {}^UE_2^{3,2p} & \xrightarrow{\Psi^*} & {}^TE_2^{3,2p} \\
\downarrow & & \downarrow \\
{}^UE_{2p-1}^{3,2p} & \xrightarrow{\Psi^*} & {}^TE_{2p-1}^{3,2p} \\
\downarrow {}^Ud_{2p-1} & & \downarrow {}^Td_{2p-1} \\
{}^UE_{2p-1}^{2p+2,2} & \xrightarrow{\Psi^*} & {}^TE_{2p-1}^{2p+2,2} \\
\hookdownarrow & & \hookdownarrow \\
M^2 = {}^UE_2^{2p+2,2} & \xrightarrow{\Psi^*} & {}^TE_2^{2p+2,2}
\end{array}$$

where the composition of the left vertical maps is δ^1 and we resume the computation of $\Psi^*\delta^1(c_px_1)$ started in (3.3):

$$\begin{aligned}
&\Psi^*\delta^1(c_px_1)\\
&= {}^Td_{2p-1}(\binom{n-1}{p-1}\sum_{i=1}^n v'_iv_n^{p-1}x_1) \quad \text{(by (3.4))}\\
&= \binom{n-1}{p-1}\sum_{i=1}^n v'_iy_{p,0} \quad \text{(since } v'_i\text{'s are permanent cocycles)}\\
&= \binom{n-1}{p-1}\sum_{i=1}^n v_iy_{p,0} \quad \text{(since } y_{p,0} \text{ is } p\text{-torsion)}\\
&= \Psi^*(\binom{n-1}{p-1}c_1y_{p,0}).
\end{aligned} \tag{3.5}$$

By the injectivity of

$$\Psi^*: M^2 = {}^U E_2^{2p+2,2} \to {}^T E_2^{2p+2,2}$$

together with (3.5), we have

$$\delta^1(c_p x_1) = \binom{n-1}{p-1} c_1 y_{p,0} \neq 0$$

and we conclude. □

Lemma 3.3. *The chain complex $\mathcal{M}$ is exact at M^2.*

Proof. By Lemma 3.2, and the fact that $\mathcal{M}$ is a chain complex, we have $\delta^2 = 0$ and the lemma follows.

Alternatively, one may compute $\delta^2 = d_3^{2p+2,2}$ directly with Corollary 2.9 and obtain the same result. □

The exactness of $\mathcal{M}$ at M^1. Recall that the $\mathbb{Z}_{(p)}$-module M^1 is freely generated by elements of the form cx_1 for

$$c \in S' := \{c_1^{i_1} c_2^{i_2} \cdots c_p^{i_p} \mid i_k \geq 0,\ \sum_k k i_k = p\}.$$

Indeed, S' is simply the set of monomials in $c_1, c_2 \cdots, c_n$ in $H^{2p}(BU_n)$. We define a total ordering $\mathfrak{O}$ on monomials in $c_1, c_2 \cdots, c_n$ as follows. We assert

$$c_1^{i_1} c_2^{i_2} \cdots c_p^{i_p} > c_1^{j_1} c_2^{j_2} \cdots c_p^{j_p}$$

if and only if

(1) there is at least one k such that $i_k \neq j_k$, and
(2) for the smallest such k, we have $i_k > j_k$.

Let $S := S' - \{c_p\}$. Then $\mathfrak{O}$ defines total orderings on S, S' and $S'x_1$ as well. To compare $cx_1, c'x_1 \in S'x_1$, we assert $cx_1 > c'x_1$ if and only if $c > c'$.

Let L be the $\mathbb{Z}_{(p)}$-submodule of $H^{2p}(BU_n)_{(p)}$ spanned by S. We define a $\mathbb{Z}_{(p)}$-linear map

$$\tau : L \to M^0 = H^{2p+2}(BU_n)_{(p)}$$

as follows. Each element in S is of the form $c_1^{i_1} c_2^{i_2} \cdots c_k^{i_k}$ such that $k < p$ and $i_k > 0$, and we define

$$\tau(c_1^{i_1} c_2^{i_2} \cdots c_k^{i_k}) := (c_1^{i_1} c_2^{i_2} \cdots c_{k-1}^{i_{k-1}})(c_k^{i_k - 1} c_{k+1}).$$

Lemma 3.4. *Let $\bar{\tau} : L/pL \to M^0/pM^0$ and $\bar{\delta}^0 : M^0/pM^0 \to M^1/pM^1$ denote the mod p reductions of τ and δ^0, respectively. Then the image of the composition*

$$L/pL \xrightarrow{\bar{\tau}} M^0/pM^0 \xrightarrow{\bar{\delta}^0} M^1/pM^1$$

is Lx_1/pLx_1. In particular, we have

$$\operatorname{Im} \delta^0 \tau \subset W := Lx_1 + (pc_p x_1) \subset M^1. \tag{3.6}$$

Proof. Consider the $\mathbb{Z}_{(p)}$-basis S, $S'x_1$ for L and M^1, respectively, both in the descending order with respect to the ordering $\mathfrak{O}$. Notice that $c_p x_1$ is the smallest element in S'. We study the $(N+1) \times N$ matrix A of the map

$$\delta^0 \tau : L \to M^1$$

with respect to these basis, where N is the cardinality of S.

Consider an arbitrary element

$$c := c_1^{i_1} \cdots c_k^{i_k} \in S$$

with $k < p$ and $i_k > 0$. By Corollary 2.9 and the Leibniz's formula, we have

$$\delta^0\tau(c) = \delta^0(c_1^{i_1} \cdots c_{k-1}^{i_{k-1}} c_k^{i_k - 1} c_{k+1})$$
$$= \begin{cases} (n-k)cx_1 + ni_1 c_1^{i_1-1} c_2^{i_2} \cdots c_k^{i_k-1} c_{k+1} x_1 + (\text{higher order terms}), \ i_1 > 0, \\ (n-k)cx_1 + (\text{higher order terms}), \ i_1 = 0. \end{cases}$$

In both cases, we have

$$\delta^0\tau(c) \equiv (n-k)cx_1 + (\text{higher order terms}) \pmod p.$$

Therefore, the matrix A satisfies

$$A \equiv \begin{pmatrix} \lambda_1 & * & \cdots & * \\ 0 & \lambda_2 & * & \vdots \\ 0 & 0 & \ddots & * \\ 0 & \cdots & 0 & \lambda_N \\ 0 & 0 & \cdots & 0 \end{pmatrix} \pmod p,$$

where the λ_i's are of the form $n-k$ for $k < p$, which are invertible in $\mathbb{Z}_{(p)}$, and we have verified that the image of the composition

$$L/pL \xrightarrow{\bar{\tau}} M^0/pM^0 \xrightarrow{\bar{\delta}^0} M^1/pM^1$$

is Lx_1/pLx_1. The equation (3.6) follows from the above and the fact

$$M^1 = Lx_1 + (c_p x_1).$$

□

Lemma 3.5. *Consider the $\mathbb{Z}_{(p)}$-submodule $V = \tau(L) + (c_1c_p - c_{p+1})$ of M^0. We have $\delta^0(V) \subset W$ where*

$$W := Lx_1 + (pc_p x_1) \subset M^1$$

is the $\mathbb{Z}_{(p)}$-submodule of M^1 defined in Lemma 3.4.

Proof. By Lemma 3.4 we have $\delta^0(\tau(L)) \subset W$. On the other hand, we have

$$\delta^0(c_1c_p - c_{p+1}) = (n-p+1)c_1c_{p-1}x_1 + pc_px_1 \in W, \tag{3.7}$$

and we conclude. □

Lemma 3.6. *The chain complex $\mathcal{M}$ is exact at M^1.*

Proof. By Lemma 3.5, the restriction of δ^0 to V has image in W. Therefore, we write $\delta_V^0 := \delta^0|_V : V \to W$ and consider its mod p reduction

$$\bar{\delta}_V^0 : V/pV \to W/pW = Lx_1/pLx_1 + (pc_px_1)/(p^2c_px_1).$$

By Lemma 3.4, we have $Lx_1/pLx_1 \subset \operatorname{Im}\bar{\delta}^0|_V$.

By $Lx_1/pLx_1 \subset \operatorname{Im}\bar{\delta}_V^0$ and (3.7), we have $[pc_px_1] \in \operatorname{Im}\bar{\delta}_V^0$, where $[pc_px_1]$ is the class in W/pW represented by pc_px_1. Therefore, $\bar{\delta}_V^0 : V/pV \to W/pW$ is surjective. By Nakayama's lemma in commutative algebra (Thoerem 2.2, Chapter 1, [17]), $\delta_V^0 : V \to W$ is surjective.

Therefore, we have

$$\operatorname{Im}\delta^0 \supset \operatorname{Im}\delta_V^0 = W = Lx_1 + (pc_px_1). \tag{3.8}$$

On the other hand, we have $\operatorname{Ker}\delta^1 \supset \operatorname{Im}\delta^0$, and therefore $\operatorname{Ker}\delta^1 \supset W$. Now, by Lemma 3.2, we have

$$\mathbb{Z}/p \cong M^1/(L + (pc_p x_1)) = M^1/W \to M^1/\operatorname{Ker}\delta^1 \cong \mathbb{Z}/p,$$

where the arrow is the tautological quotient map, which is surjective. Therefore, the above composition is a bijection. It follows that we have

$$W = \operatorname{Ker}\delta^1 \supset \operatorname{Im}\delta^0, \tag{3.9}$$

and the lemma follows from (3.8) and (3.9). □

Lemma 3.6 and Lemma 3.3 complete the proof of Proposition 2.10.

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