

Math 2177 recitation: PDE 2

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(You can find all my recitation handouts and their solutions on my homepage <http://u.osu.edu/yuzhang/teaching/>)

1 Solving heat equation

As an example, we look at the following partial differential equation (PDE):

$$\begin{cases} u_t = \beta u_{xx}, & 0 < x < L, \quad t > 0 & \text{(PDE)} \\ u(0, t) = u(L, t) = 0, & t > 0 & \text{(Boundary Condition)} \\ u(x, 0) = f(x), & 0 < x < L & \text{(Initial Condition)} \end{cases}$$

By separating variables, we can solve this PDE in 4 steps:

Step 1. Write $u(x, t) = X(x)T(t)$ to turn the PDE into two ordinary differential equations (with boundary conditions)

Let $u(x, t) = X(x)T(t)$, we obtain the boundary value problem

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = X(L) = 0 \end{cases} \quad \text{and } T'(t) = -\lambda\beta T(t)$$

Step 2. Find all eigenvalues λ_n and their corresponding eigenfunctions X_n of the boundary value problem in step 1.

Depending on the value of λ , the boundary value problem

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = X(L) = 0 \end{cases}$$

may only have zero solution $X(x) \equiv 0$. We want to determine those values of λ for which the boundary value problem has nontrivial solutions. These nontrivial solutions are called the eigenfunctions of the problem, the eigenvalues are those corresponding values of λ .

By computations we conclude eigenvalues are $\lambda_n = (\frac{n\pi}{L})^2$ and corresponding eigenfunctions are $X_n(x) = \sin(\frac{n\pi}{L}x)$.

Step 3. Use λ_n to find corresponding T_n . Then find the general solution $u(x, t) = \sum_{n=1}^{\infty} c_n X_n(x)T_n(t)$ satisfying both the PDE and boundary condition.

For $\lambda = (\frac{n\pi}{L})^2$, general solution of $T'(t) = -\beta\lambda T(t)$ is

$$T_n(t) = ce^{-\beta\lambda t} = ce^{-\beta(\frac{n\pi}{L})^2 t}$$

Now general solution for $u(x, t)$ is

$$u(x, t) = \sum_{n=1}^{\infty} c_n X_n(x) T_n(t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L}x\right) e^{-\beta\left(\frac{n\pi}{L}\right)^2 t}$$

Step 4. Use the initial condition to determine the coefficients c_n then get final answer $u(x, t)$.

From the initial condition $u(x, 0) = f(x)$ we know $\sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L}x\right) = f(x)$. When $f(x)$ is already a linear combination of $\sin\left(\frac{n\pi}{L}x\right)$, we can directly read the coefficients c_n and get final answer $u(x, t)$

Exercise 1. Find the solution to the heat flow problem

$$\begin{cases} u_t = 7u_{xx}, & 0 < x < \pi, \quad t > 0 \\ u(0, t) = u(\pi, t) = 0, & t > 0 \\ u(x, 0) = 3\sin(2x) - 6\sin(5x), & 0 < x < \pi \end{cases}$$

Exercise 2. Find the solution to the heat flow problem

$$\begin{cases} u_t = 2u_{xx}, & 0 < x < 1, \quad t > 0 \\ u(0, t) = u(1, t) = 0, & t > 0 \\ u(x, 0) = 3\sin(3\pi x) + 5\sin(5\pi x) + \sin(9\pi x), & 0 < x < 1 \end{cases}$$

2 Fourier series

Let $f(x)$ be a continuous periodic function with period $2L$. Then $f(x)$ has a Fourier series

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L}))$$

where

$$a_0 = \frac{1}{L} \int_{-L}^L f(x) dx$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos(\frac{n\pi x}{L}) dx$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin(\frac{n\pi x}{L}) dx$$

Moreover when $f(x)$ and $f'(x)$ are piecewise continuous, $F(x) = f(x)$ for all x .

Special cases:

(1) If $f(x)$ is an even function meaning $f(x) = f(-x)$, then all $b_n = 0$.

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{L})$$

(2) If $f(x)$ is an odd function meaning $f(x) = -f(-x)$, then all $a_n = 0$.

$$F(x) = \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{L})$$

Exercise 3. Consider the following function

$$f(x) = \pi^2 - x^2, -\pi \leq x \leq \pi, f(x + 2\pi) = f(x)$$

(a) Is $f(x)$ even, odd, or neither?

(b) Find the Fourier series $F(x)$ of the given $f(x)$ with period $T = 2\pi$. You may use the information you obtain in (a).

(c) What is $F(\pi)$ and $F(4\pi)$?

(d) Evaluate $\sum_{n=1}^{\infty} \frac{1}{n^2}$ using parts (b) and (c), i.e., the expression of $F(\pi)$ as a series when $x = \pi$ and the value of $F(\pi)$ from the convergence theorem.