Math 2177 midterm 2 review exercises

TA: Yu Zhang

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1 Solve systems of equations

Exercise1 : Solve the following systems of linear equations:

(1) $\begin{cases} x_1 + x_2 - 2x_3 = 1 \\ 2x_1 - 3x_2 + x_3 = -8 \\ 3x_1 + x_2 + 4x_3 = 7 \end{cases}$ (2) $\begin{cases} x_1 - x_2 + x_3 - x_4 = 2 \\ x_1 - x_2 + x_3 + x_4 = 0 \\ 4x_1 - 4x_2 + 4x_3 = 4 \\ -2x_1 + 2x_2 - 2x_3 + x_4 = -3 \end{cases}$

Solution : (1) $\begin{cases} x_1 = 0 \\ x_2 = 3 \\ x_3 = 1 \end{cases}$

(2) $\begin{cases} x_1 = 1 + s - t \\ x_2 = s \\ x_3 = t \\ x_4 = -1 \end{cases}$ where s, t are any real numbers.

2 Connection between homogeneous and non homogeneous equations

Exercise2 : Let $\mathbf{A} = \begin{bmatrix} 2 & 3 & -1 & -9 \\ 0 & 1 & 1 & 1 \\ -1 & 2 & 3 & 4 \end{bmatrix}$.

(1) Find all solutions to $\mathbf{A}\overline{x} = 0$

(2) Find all solutions to $\mathbf{A}\overline{x} = \overline{b}$ given that $\overline{p} = \begin{bmatrix} 3 \\ -5 \\ 7 \\ 0 \end{bmatrix}$ is a solution to $\mathbf{A}\overline{x} = \overline{b}$.

Describe the solutions in parametric vector form, and give a geometric description of the solution sets.

Solution : (1) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2t \\ 3t \\ -4t \\ t \end{bmatrix}$, where t is any real number.

(2) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \\ 7 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 3 \\ -4 \\ 1 \end{bmatrix}$, where t is any real number.

Geometrically, it is a line in R^4 passing through $\begin{bmatrix} 3 \\ -5 \\ 7 \\ 0 \end{bmatrix}$.

Exercise3 : Let $\mathbf{A} = \begin{bmatrix} 2 & 1 & 7 & -2 \\ 3 & -2 & 0 & 11 \\ 1 & 1 & 5 & -3 \end{bmatrix}$.

(1) Find all solutions to $\mathbf{A}\overline{x} = 0$

(2) Find all solutions to $\mathbf{A}\overline{x} = \overline{b}$ given that $\overline{p} = \begin{bmatrix} 3 \\ -2 \\ 0 \\ 0 \end{bmatrix}$ is a solution to $\mathbf{A}\overline{x} = \overline{b}$.

Describe the solutions in parametric vector form, and give a geometric description of the solution sets.

Solution : (1) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2s - t \\ -3s + 4t \\ s \\ t \end{bmatrix}$, where s, t are any real numbers.

(2) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 4 \\ 0 \\ 1 \end{bmatrix}$, where s, t are any real numbers.

Geometrically, it is a plane in R^4 passing through $\begin{bmatrix} 3 \\ -2 \\ 0 \\ 0 \end{bmatrix}$.

3 Matrix operations

Exercise4 : Compute $\begin{bmatrix} 2 & 5 \\ -1 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 & 5 & -3 & 4 \\ 2 & 0 & 2 & -3 \end{bmatrix}$

Solution : $\begin{bmatrix} 12 & 10 & 4 & -7 \\ 5 & -5 & 9 & -13 \\ -2 & 10 & -10 & 14 \end{bmatrix}$

Exercise5 : Let $A = \begin{bmatrix} 1 & 2 & 3 & -2 \\ 0 & 1 & -2 & -1 \\ 1 & 1 & 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -7 \\ 3 \\ 1 \\ 1 \end{bmatrix}$ Compute the following matrix operations. Write "undefined" for expressions that are undefined.

(1)A^T (2)B^T (3)AB (4)$A^T B$ (5)AB^T (6)$A^T B^T$

Solution : (1) $A^T = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 3 & -2 & 3 \\ -2 & -1 & 1 \end{bmatrix}$ (2) $B^T = \begin{bmatrix} -7 & 3 & 1 & 1 \end{bmatrix}$

(3)$AB = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (4)Undefined (5) Undefined (6)Undefined

Exercise6 : Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. Compute A^2, A^3, A^4, A^5.

Solution : $A^2 = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$, $A^3 = \begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix}$, $A^4 = \begin{bmatrix} 1 & 8 \\ 0 & 1 \end{bmatrix}$, $A^5 = \begin{bmatrix} 1 & 10 \\ 0 & 1 \end{bmatrix}$.

4 Linear combination, linearly dependence and singular matrix

Exercise7 : (1) Let $v_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 4 \end{bmatrix}$, $v_2 = \begin{bmatrix} 3 \\ 2 \\ -2 \\ 1 \end{bmatrix}$, $w = \begin{bmatrix} 5 \\ 8 \\ -12 \\ -5 \end{bmatrix}$. Determine whether w is a linear combination of v_1 and v_2.

(2) Determine whether v_1, v_2 and w are linearly dependent.

Solution :

$$(1) \begin{bmatrix} 2 & 3 & 5 \\ -1 & 2 & 8 \\ 3 & -2 & -12 \\ 4 & 1 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$w = -2v_1 + 3v_2$ is a linear combination of v_1 and v_2.

(2) From (1), we know $w = -2v_1 + 3v_2$. Therefore $-2v_1 + 3v_2 - w = w - w = 0$ is a nonzero solution to $x_1v_1 + x_2v_2 + x_3w = 0$. v_1, v_2 and w are linearly dependent.

Exercise8 : (1) Let $A = \begin{bmatrix} 2 & 1 & 3 & 1 \\ -1 & 1 & -1 & 1 \\ 3 & 2 & 0 & 5 \\ 0 & -1 & -2 & 1 \end{bmatrix}$ Determine whether A is singular.

(2) Let $v_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ -1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 3 \\ -1 \\ 0 \\ -2 \end{bmatrix}$, $v_4 = \begin{bmatrix} 1 \\ 1 \\ 5 \\ 1 \end{bmatrix}$. Determine whether v_1, v_2 , v_3 and v_4 are linearly dependent.

Solution :

(1) $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ has pivot in each column. A is non singular.

(2) From (1) A has pivots in each column so $x_1v_1 + x_2v_2 + x_3v_3 + x_4v_4 = 0$ only has zero solution. v_1, v_2 , v_3 and v_4 are linearly independent.

Exercise9 : Let $\{v_1, v_2, v_3\}$ be linearly independent vectors. Determine whether $\{2v_1 + 3v_2 + v_3, v_1 - v_2 + 2v_3, 2v_1 + v_2 - v_3\}$ is linearly dependent.

Solution :

Consider the equation

$$x_1(2v_1 + 3v_2 + v_3) + x_2(v_1 - v_2 + 2v_3) + x_3(2v_1 + v_2 - v_3) = 0$$

We can rearrange this equation to

$$(2x_1 + x_2 + 2x_3)v_1 + (3x_1 - x_2 + x_3)v_2 + (x_1 + 2x_2 - x_3)v_3 = 0$$

Since $\{v_1, v_2, v_3\}$ is linearly independent, we must have $\begin{cases} 2x_1 + x_2 + 2x_3 = 0 \\ 3x_1 - x_2 + x_3 = 0 \\ x_1 + 2x_2 - x_3 = 0 \end{cases}$

Solving this system of equations shows $x_1 = x_2 = x_3 = 0$. Therefore $\{2v_1 + 3v_2 + v_3, v_1 - v_2 + 2v_3, 2v_1 + v_2 - v_3\}$ is linearly independent.