Math 2177 recitation: row (reduced) echelon form

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1 Row (reduced) echelon form

We say a matrix is in **row echelon form** (REF) if

1. All nonzero rows (rows with at least one nonzero element) are above any rows of all zeroes (all zero rows, if any, belong at the bottom of the matrix)

2. The leading coefficient (the first nonzero number from the left, also called the pivot) of a nonzero row is always strictly to the right of the leading coefficient of the row above it

Examples:

$$\begin{bmatrix}1 & 5 & 1 & 2\\0 & 1 & 0 & 7\\0 & 0 & 4 & 9\end{bmatrix} \begin{bmatrix}1 & 2 & 3\\0 & 4 & 5\\0 & 0 & 0\end{bmatrix} \begin{bmatrix}0 & 0 & 3 & 1 & 2 & 0 & 4\\0 & 0 & 0 & 5 & 12 & 7 & 0\\0 & 0 & 0 & 0 & 0 & 0 & 2\end{bmatrix}$$

We say a matrix is in **row reduced echelon form** (RREF) if

1. It is in row echelon form.
2. The leading entry in each nonzero row is a 1 (called a leading 1).
3. Each column containing a leading 1 has zeros everywhere else.

Examples:

$$\begin{bmatrix}1 & 0 & 3 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 1\end{bmatrix} \begin{bmatrix}1 & 0 & 0\\0 & 1 & 1\\0 & 0 & 0\end{bmatrix} \begin{bmatrix}0 & 1 & 1 & 3 & 0 & 0 & 12\\0 & 0 & 0 & 0 & 1 & 0 & 9\\0 & 0 & 0 & 0 & 0 & 1 & 7\end{bmatrix}$$

2 Method

A matrix can be changed to its row (reduced) echelon form using elementary row operations:

A. Interchange one row of the matrix with another of the matrix.
B. Multiply one row of the matrix by a nonzero constant.
C. Replace the one row with the one row plus a constant times another row of the matrix.

To reduce a matrix into **row echelon form**:

1. Find leftmost nonzero column.
2. Find a nonzero number in this column. Interchange rows to make it in first row.
3. Create zeros in all positions below that pivot.
4. Repeat step 1-3 for the sub matrix to the bottom right of that pivot.

To further reduce a **row echelon form** matrix into **row reduced echelon form**:

5. For each pivot, multiply that row by a constant to make pivot equal to 1.
6. Create zeros above each leading 1 starting from the rightmost pivot.

3 Examples

Example: Reduce the following matrices into row echelon form. Then further reduce to reduced echelon form.

(1) $\begin{bmatrix}2 & 6 & 0\\1 & 2 & 1\end{bmatrix}$

Row echelon form: $\begin{bmatrix}2 & 6 & 0\\1 & 2 & 1\end{bmatrix} \sim \begin{bmatrix}1 & 3 & 0\\1 & 2 & 1\end{bmatrix} \sim \begin{bmatrix}1 & 3 & 0\\0 & -1 & 1\end{bmatrix}$

Row reduced echelon form: $\begin{bmatrix}1 & 3 & 0\\0 & -1 & 1\end{bmatrix} \sim \begin{bmatrix}1 & 3 & 0\\0 & 1 & -1\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 3\\0 & 1 & -1\end{bmatrix}$

(2) $\begin{bmatrix} 1 & 2 & 0 \\ 1 & 3 & 3 \\ -1 & 0 & -1 \\ -3 & 0 & 0 \end{bmatrix}$

Row echelon form: $\begin{bmatrix} 1 & 2 & 0 \\ 1 & 3 & 3 \\ -1 & 0 & -1 \\ -3 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ -1 & 0 & -1 \\ -3 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 2 & -1 \\ -3 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 2 & -1 \\ 0 & 6 & 0 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & -7 \\ 0 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & -7 \\ 0 & 0 & -18 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & -7 \\ 0 & 0 & 0 \end{bmatrix}$

Row reduced echelon form: $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & -7 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(3) $\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$

Row echelon form: $\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix} \sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$

$\sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$

Row reduced echelon form: $\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -3 & 4 & -3 & 0 & -3 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$